Structural Aspects in the Theory of Probability

Second Enlarged Edition

Herbert Heyer

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SERIES ON MULTIVARIATE ANALYSIS

Editor: M M Rao

Published

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Cover design by Katrin Heyer

Printed in Singapore.

Preface
to the second enlarged edition

The appearence of the present second edition is motivated by a two-fold engagement: to improve parts of the original text and to add two chapters on recent developments.

The improvements include the clarification of obscurities and the correction of formal flaws. The additional chapters serve as useful supplements to the topics treated in the first edition. The discussion added to Section 5.4 concerns the embedding of infinitely divisible probability measures on an Abelian locally compact group, a problem deeply related to the structure of the underlying group.

The new Chapter 7 is devoted to extending the framework of an Abelian group to that of a commutative hypergroup. Chapter 8 written by Gyula Pap aims at the central limit problem of probability theory for Abelian groups. While this theme enriches the previous discussion of the asymptotic behavior of random sequences in groups, the approach to stochastic processes taken in Chapter 7 reaches beyond groups, i.e. to group-like structures defined via a generalized convolution. In both chapters limit theorems for random sequences are discussed with respect to the special structure of their state space, particularly for tori, p-adic groups and solenoids in Chapter 8, for polynomial and Sturm–Liouville structures in Chapter 7.

Although the technique employed in both chapters is based on the notion of the Fourier transform —as throughout the first edition—, hypergroups require special attention. Meeting this requirement within a reasonably sized text lead necessarily to an expository presentation in Chapter 7. The main idea for choosing to study infinitesimal arrays in Chapter 8 is the completion and extension of the only available reference to necessary and sufficient conditions in terms of moments of the convergence of such arrays towards weakly divisible distributions.

In Chapter 7 the author prepares the necessary tools for the construction of hypergroup structures and their analysis before he enters the treatment of random walks and more general increment processes in these structures. Despite obvious similarities to the group case significant deviations justify the extended concept and open the route to further fruitful research.

The author is grateful to his publisher for having stimulated this second enlarged edition of his book.

Heartfelt thanks go to two of his colleagues and friends: to Michael Bingham, who proposed numerous ameliorations, and to Gyula Pap, who not only adapted the original typescript to the requirements of the publisher, but also added significant improvements and above all composed Chapter 8 on the basis of his recent research.

As usual and most remarkably again the team of World Scientific has cooperated optimally along the production of the present edition.

And, as not even multi-editions are free from dark spots, all future readers are invited to report on their concerns.

Tübingen, June 2009   Herbert Heyer

Preface

The present book has been written for mathematically prepared readers who like to look beyond the boundary of a single topic in order to discover the interrelations with others. More concretely the author's idea is to direct the attention of probabilists to the applicability of the enlightening notion of a group to probability theory.

The interplay between probability theory and group theory is as old as the early investigations on translation invariant probability distributions and stochastic processes and has become an increasingly important field of research which meanwhile reached a certain state of maturity.

While the traditional approach to the basic theorems of probability theory often overshadows part of the structure of the problems, the awareness of group-theoretical concepts leads to a quick detection of common features of apparently unrelated situations. In other words, the perception of algebraic-topological structures in the state space of stochastic processes does not only yield interesting and applicable generalizations of known results but also sets a limit to such generalizations by describing their domains of validity within the general framework. In practice this approach helps to provide at least more transparent proofs of well-established theorems including Lévy's continuity theorem, the Lévy–Khinchin representation of infinitely divisible probability measures, transience criteria for convolution semigroups and characterizations of recurrent or transient random walks.

This primer in probabilities on Abelian topological groups with emphasis on separable Banach spaces and on locally compact Abelian groups is by its very conception an elementary introduction to the structural access to probability theory, no textbook in the habitual understanding and by now means a monograph. It should be studied by graduate students along with the course work and will make interesting accompanying reading for

their lecturers. At the same time the book provides information beyond the particular topic and lays bare the possibility of incorporating certain problems of probability theory into a wider setting which may be chosen according to the actual aims of study.

Since the pioneering work of Grenander and Parthasarathy going back to the early 1960's structural aspects of probability theory have been stressed in various monographs. For probabilities on locally compact groups we mention the books by Berg and Forst and by Revuz, both of 1975, as well as the author's book of 1977. There is also an extensive literature on probabilities on linear spaces. We just cite the books by Araujo and Giné of 1980, by Linde of 1986 and by Vakhania, Tarieladze and Chobanyan of 1987. Our selection of topics from these sources has at least two motives: to stress the significance of the problems within the development of the theory, and to choose an approach to their solutions which at the same time is as direct and informative as possible. Clearly these aims can hardly be achieved without reference to some basic notions and facts from topological groups, topological vector spaces and commutative Banach algebras. Appendices at the end of the book are offered as desirable aids.

In the first part of the book (Chapters 1 to 3) we start by collecting the necessary measure theory on metric spaces including the Riesz and Prohorov theorems. It follows a detailed analysis of the Fourier transform for separable Banach spaces. The main focus of the subsequent discussion is the arithmetic of probability measures on such spaces, in particular the study of infinitely divisible probability measures. We establish the embedding of infinitely divisible probability measures into continuous convolution semigroups and then examine Gauss and Poisson measures. The Ito–Nisio theorem is applied to a construction of Brownian motion. The proof of the Lévy–Khinchin representation is prepared by a detailed discussion of Lévy measures and generalized Poisson measures. It is clear that the theory exposed for general separable Banach spaces covers the case of Euclidean space and also various cases of function spaces.

The second half of the book (Chapters 4 to 6) begins with the notion of convolution of Radon measures on a locally compact group. The exposition continues by developing the duality theory of locally compact Abelian groups including positive definite functions and measures. Then negative definite functions on such groups are studied, their duality with positive definite functions and their correspondence in the sense of Schoenberg with convolution semigroups. The construction of Lévy functions for any locally compact Abelian group is the basic step towards a Lévy–Khinchin repre-

sentation of negative definite functions. The concluding chapter contains a discussion of transient convolution semigroups and random walks. A measure-theoretic proof of the Port–Stone transience criterion precedes the characterization of groups admitting recurrent random walks and the classification of transient random walks which solves the problem of renewal of random walks on a locally compact Abelian group. The theory developed in this part of the book can be easily specialized to the Euclidean case, but moreover to infinite dimensional lattices and tori.

Now the methodical framework of the book becomes visible. For separable Banach spaces as well as for locally compact Abelian groups dual objects and Fourier transforms of measures as functions on these dual objects are employed in order to determine the structure of infinitely divisible probability measures and convolution semigroups. For Banach spaces only restricted versions of the Lévy continuity theorem can be proved. In fact, by the lack of an appropriate Bochner theorem for positive definite functions harmonic analysis soon reaches its limits. In the case of locally compact Abelian groups, however, the Pontryagin duality provides a far more elaborate harmonic analysis which can be applied to obtain not only strong versions of the Lévy continuity theorem but also deep results on the potential theory of stochastic processes with stationary independent increments and random walks in the group.

To write a primer in probabilities on algebraic-topological structures became a matter of concern during the author's lecturing over about three decades, mostly at the University of Tübingen in Germany. Along with his research work at the interface between probability theory and harmonic analysis he taught on probability measures on Banach spaces, locally compact groups and homogeneous spaces. It turned out that graduate students majoring in probability theory or in analysis took those courses which led to seminars on "Stochastics and Analysis" in which central limit theorems for generalized random variables, stochastic processes in and random fields over general algebraic-topological structures were discussed. In recent years also analogs of these probabilistic objects for generalized convolution structures as Jacobi and Sturm–Liouville translation structures were considered. For the harmonic analysis of these structures the presentation of the case of a locally compact Abelian group provides the appropriate basis. Consequently, the present book may also be used as a preparatory text for the study of probability measures on hypergroups and hypercomplex systems.

In conceiving his book the author received encouragement from many colleagues and friends spread over the globe. Various scientific agencies

like the German and the Japanese Research Societies made it possible to test preliminary versions of the manuscript in workshops and crash courses during research stays and sabbaticals at universities in Australia (Perth), Japan (Tokyo) and the US (San Diego). Acknowledgement of prime importance goes to Christian Berg and Gunnar Forst, to Werner Linde and to Daniel Revuz for their excellent monographs the contents of which reaches far beyond our exposition. Several people have read drafts of the text. Especially valuable was Gyula Pap's constructive criticism for which the author is most thankful. There were also capable secretaries who did a great job in preparing the typescript: Kerstin Behrends and Erika Gugl deserve praise for their skillful work. Last but not least I am grateful to M.M. Rao from the University of California at Riverside who invited the book into the series on Multivariate Analysis with World Scientific.

The author expresses his expectation that all obscurities contained in the text will be communicated to him and that despite such inevitable deficiencies the book may serve its modest purpose. There is no doubt that the following statement due to Pablo Picasso also applies to an author in mathematics

"Ce que je fais aujourd'hui est déja vieux pour demain."

Tübingen, March 2004   Herbert Heyer

Contents

Chapter 1

Probability Measures on Metric Spaces

1.1 Tight measures

Let (E, d) denote a metric space, $\mathcal{O}(E)$, $\mathcal{A}(E)$, $\mathcal{K}(E)$ the systems of open, closed and compact subsets of E respectively. On (E, d) we have the notions of the *Borel σ-algebra*

$$\mathfrak{B}(E) := \sigma(\mathcal{O}(E)) = \sigma(\mathcal{A}(E))$$

of E and of a (Borel) measure on E, i.e. a non-negative extended real-valued σ-additive set function μ on $\mathfrak{B}(E)$ with the properties that $\mu(\emptyset) = 0$ and $\mu(K) < \infty$ for all $K \in \mathcal{K}(E)$.

Definition 1.1.1 A finite measure μ on E is called

(a) **regular** if for every $B \in \mathfrak{B}(E)$ and for every $\varepsilon > 0$ there exist $A \in \mathcal{A}(E)$ and $O \in \mathcal{O}(E)$ such that $A \subset B \subset O$ and $\mu(O) - \mu(A) < \varepsilon$,

and

(b) **tight** if

$$\mu(E) = \sup\{\mu(K) : K \in \mathcal{K}(E)\}.$$

Theorem 1.1.2 *Let μ be a finite measure on E. Then*

(i) *μ is regular.*

(ii) *If μ is tight then it must be **inner-regular** in the sense that for each $B \in \mathfrak{B}(E)$*

$$\mu(B) = \sup\{\mu(K) : K \in \mathcal{K}(E), \ K \subset B\}.$$

In particular, for finite measures the notions of tightness and inner-regularity coincide.

Proof. (i). Let $\mathfrak{D} := \mathfrak{D}_\mu$ be the system of all $B \in \mathfrak{B}(E)$ with respect to which μ is regular. Then $\mathfrak{D}$ is a Dynkin system in the sense that $E \in \mathfrak{D}$, $B \in \mathfrak{D}$ implies that $\complement B \in \mathfrak{D}$, and whenever $(B_n)_{n \geqslant 1}$ is a disjoint sequence in $\mathfrak{D}$ then $B := \bigcup_{n \geqslant 1} B_n \in \mathfrak{D}$.

The proof of the first property is clear, and for the second one we observe that if $A \in \mathcal{A}(E)$ and $O \in \mathcal{O}(E)$ are chosen as in Definition 1.1.1 (a) then $\complement O \subset \complement B \subset \complement A$ and, noting that $\complement O \in \mathcal{A}(E)$ and $\complement A \in \mathcal{O}(E)$, we have

$$\mu(\complement A) - \mu(\complement O) = \mu(O) - \mu(A) < \varepsilon.$$

As for the third property, given $\varepsilon > 0$ we can find $A_n \in \mathcal{A}(E)$ and $O_n \in \mathcal{O}(E)$ with $A_n \subset B_n \subset O_n$ and

$$\mu(O_n) - \mu(A_n) < \frac{1}{2^{n+2}} \varepsilon$$

for all $n \in \mathbb{N}$. Let $O := \bigcup_{n \geqslant 1} O_n$, choose n_0 with $\mu\left(\bigcup_{n > n_0} A_n\right) < \varepsilon/4$ and put $A := \bigcup_{n=1}^{n_0} A_n$. Then $A \in \mathcal{A}(E)$, $O \in \mathcal{O}(E)$, $A \subset B \subset O$ and

$$\mu(O \setminus A) \leqslant \sum_{n \geqslant 1} \mu(O_n \setminus A) \leqslant \sum_{n=1}^{n_0} \mu(O_n \setminus A) + \sum_{n > n_0} \mu(O_n)$$

$$\leqslant \frac{\varepsilon}{4} + \sum_{n > n_0} \left(\mu(A_n) + \frac{1}{2^{n+2}} \varepsilon\right) \leqslant \frac{3}{4} \varepsilon < \varepsilon.$$

Furthermore $\mathcal{A}(E) \subset \mathfrak{D}$. Indeed, given $A \in \mathcal{A}(E)$ for each $n \in \mathbb{N}$ we observe that

$$A^{\frac{1}{n}} := \left\{x \in E : d(x, A) < \frac{1}{n}\right\}$$

is open, and from $A^{\frac{1}{n}} \downarrow A$ (which holds as E is metric) it follows that $\mu(A^{\frac{1}{n}}) \downarrow \mu(A)$.

Now $\mathcal{A}(E)$ is $\cap$-stable, and therefore

$$\mathfrak{B}(E) = \sigma(\mathcal{A}(E)) = \mathfrak{D}(\mathcal{A}(E)) \subset \mathfrak{D} \subset \mathfrak{B}(E),$$

whence $\mathfrak{D} = \mathfrak{B}(E)$. Here $\mathfrak{D}(\mathcal{A}(E))$ denotes the Dynkin hull of $\mathcal{A}(E)$.

(ii). Let $B \in \mathfrak{B}(E)$ and $\varepsilon > 0$. Using (i) there exists $A \in \mathcal{A}(E)$ with $A \subset B$ such that $\mu(B) - \mu(A) < \varepsilon/2$, and also $K \in \mathcal{K}(E)$ with $\mu(E) - \mu(K) < \varepsilon/2$. Then $A \cap K$ is a compact subset of B, and

$$\mu(B) - \mu(A \cap K) \leqslant \mu(B \setminus A) + \mu(\complement K) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

Corollary 1.1.3 *If μ is tight then for every downward filtered family $(A_\iota)_{\iota \in I}$ in $\mathcal{A}(E)$*

$$\mu\left(\bigcap_{\iota \in I} A_\iota\right) = \inf_{\iota \in I} \mu(A_\iota).$$

Proof. From $A := \bigcap_{\iota \in I} A_\iota \subset A_\kappa$ we have $\mu(A) \leqslant \mu(A_\kappa)$ for all $\kappa \in I$, and hence $\mu(A) \leqslant \inf_{\iota \in I} \mu(A_\iota)$.

In the reverse direction, appealing to Theorem 1.1.2 (ii), to each $\varepsilon > 0$ there exists $K \in \mathcal{K}(E)$ with $K \subset \complement A$ such that

$$\mu(\complement K) - \mu(A) = \mu(\complement A) - \mu(K) < \varepsilon.$$

Now $K \subset \bigcup_{\iota \in I} \complement A_\iota$, and hence by compactness there exist $\iota_1, \iota_2, \ldots, \iota_n \in I$ with $K \subset \bigcup_{i=1}^{n} \complement A_{\iota_i}$. Also $(\complement A_\iota)_{\iota \in I}$ is an upward filtered family, and hence there exists $\iota_0 \in I$ such that $K \subset \complement A_{\iota_0}$. From $\mu(\complement K) - \mu(A) < \varepsilon$ it follows that

$$\mu(A_{\iota_0}) \leqslant \mu(\complement K) < \mu(A) + \varepsilon$$

and so ε being arbitrary we obtain $\inf_{\iota \in I} \mu(A_\iota) \leqslant \mu(A)$. $\qquad \square$

Theorem 1.1.4 *Let μ be a tight measure on E.*

(i) *There exists a smallest closed subset A_0 of E with $\mu(A_0) = \mu(E)$.*
(ii) *A_0 is separable.*
(iii) *$A_0 = \{x \in E : \mu(U) > 0 \text{ for all open neighbourhoods } U \text{ of } x\}$.*

Proof. (i). The family

$$\{A \in \mathcal{A}(E) : \mu(A) = \mu(E)\}$$

is downward filtered, even $\cap$-stable. The result now follows from Corollary 1.1.3.

(ii). By Theorem 1.1.2 (ii) there exists a sequence $(K_n)_{n \geqslant 1}$ of compact and hence separable subsets of A_0 with $\mu(A_0) = \sup_{n \geqslant 1} \mu(K_n)$. Thus $A := (\bigcup_{n \geqslant 1} K_n)^-$ is separable and closed with $A \subset A_0$, from which it follows that

$$\mu(A) = \mu(A_0) = \mu(E)$$

and by (i), $A = A_0$ so that A_0 must be separable.

(iii). Write

$$B_0 := \{x \in E : \mu(U) > 0 \text{ for all open neighbourhoods } U \text{ of } x\}.$$

Given $x \in \complement A_0$ then $\complement A_0$ is an open neighbourhood of x with $\mu(\complement A_0) = 0$, and hence $x \in \complement B_0$. In the reverse direction given $x \in \complement B_0$ there exists an open neighbourhood U of x with $\mu(U) = 0$. Hence $\mu(\complement U) = \mu(E)$. Thus $A_0 \subset \complement U$ and hence $x \in \complement A_0$. $\qquad \square$

Definition 1.1.5 The set A_0 in Theorem 1.1.4 is called the **support** of μ and will be denoted by $\text{supp}(\mu)$.

Theorem 1.1.6 *Let (E, d) be a separable complete metric space. Then every finite measure μ on E is tight.*

Proof. Let $\{x_k : k \in \mathbb{N}\}$ be a dense subset of E. Then for each $n \in \mathbb{N}$

$$\bigcup_{k \geqslant 1} B\left(x_k, \frac{1}{n}\right)^- = E \,,$$

where

$$B(x, \delta) := \{y \in E : d(x, y) < \delta\}$$

is the open ball of radius $\delta > 0$ with centre x. Choose $\varepsilon > 0$. Then to each $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ satisfying

$$\mu\left(E \setminus \bigcup_{k=1}^{k_n} B\left(x_k, \frac{1}{n}\right)^-\right) \leqslant \frac{\varepsilon}{2^n} \,.$$

The set

$$K := \bigcap_{n \geqslant 1} \bigcup_{k=1}^{k_n} B\left(x_k, \frac{1}{n}\right)^-$$

is closed and totally bounded. From the completeness of E it follows that K is compact. Finally

$$\mu(\complement K) = \mu\left(\bigcup_{n \geqslant 1}\left(E \setminus \bigcup_{k=1}^{k_n} B(x_k, \frac{1}{n})^-\right)\right) \leqslant \sum_{n \geqslant 1} \frac{\varepsilon}{2^n} = \varepsilon \,. \qquad \square$$

Theorem 1.1.7 *Let E, F be metric spaces, and $\varphi : E \to F$ a continuous mapping. If μ is a tight measure on E then the image measure $\varphi(\mu)$ of μ under φ is tight on F.*

Proof. Since φ is a continuous mapping it must be $\mathfrak{B}(E)$–$\mathfrak{B}(F)$-measurable, and hence $\varphi(\mu)$ is a finite measure on F. Given $\varepsilon > 0$ there exists a compact subset K of E with $\mu(\complement K) < \varepsilon$. Also $\varphi(K)$ is a compact subset of F, and

$$\begin{aligned}
\varphi(\mu)(\complement \varphi(K)) &= \mu(\varphi^{-1}(\complement \varphi(K))) \\
&= \mu(\complement \varphi^{-1}(\varphi(K))) \\
&\leqslant \mu(\complement K) < \varepsilon \,.
\end{aligned}$$
$$\qquad \square$$

1.2   The topology of weak convergence

Although in the following discussion the set $M^{\mathrm{b}}(E)$ of all tight (finite Borel) measures on E and its subset $M^1(E) := \{\mu \in M^{\mathrm{b}}(E) : \mu(E) = 1\}$ of **probability measures** will remain the basic measure-theoretic objects, for some technical arguments we need a few facts on regular normed contents on E and related integrals. A **content** on E is a non-negative extended real-valued (finitely) additive set function μ on the algebra $\mathfrak{A}(\mathcal{O}(E))$ generated by $\mathcal{O}(E)$ satisfying $\mu(\emptyset) = 0$. Regular (finite) contents and probability contents on E are introduced in analogy to regular (finite) and probability measures on E.

Given a regular finite content μ on E, the μ-integral of a bounded real-valued function f on E is defined as follows. Let $\mathcal{P}$ be a partition of E consisting of finitely many pairwise disjoint sets $E_1, \ldots, E_n \in \mathfrak{A}(\mathcal{O}(E))$. We put

$$S_{\mathcal{P}} := \sum_{j=1}^{n} M_j \mu(E_j)$$

and

$$s_{\mathcal{P}} := \sum_{j=1}^{n} m_j \mu(E_j) \,,$$

where $M_j := \sup\{f(x) : x \in E_j\}$ and $m_j := \inf\{f(x) : x \in E_j\}$ for $j = 1, \ldots, n$. f is said to be μ-**integrable** if

$$\inf_{\mathcal{P}} S_{\mathcal{P}} = \sup_{\mathcal{P}} s_{\mathcal{P}} \,,$$

and in this case

$$\int f \,\mathrm{d}\mu := \inf_{\mathcal{P}} S_{\mathcal{P}}$$

is the μ-**integral** of f. Obviously every bounded continuous function f is μ-integrable, and

$$f \mapsto \int f \,\mathrm{d}\mu$$

defines a normed positive linear functional on the vector space $C^{\mathrm{b}}(E)$ of bounded continuous functions on E. Moreover, we have

Theorem 1.2.1 (F. Riesz)
There is a one-to-one correspondence

$$\mu \leftrightarrow L_\mu$$

between the set of regular finite (probability) contents μ on E and the set of bounded (normed) positive linear functionals L_μ on $C^{\mathrm{b}}(E)$ given by

$$L_\mu(f) := \int f \, \mathrm{d}\mu$$

for all $f \in C^{\mathrm{b}}(E)$.

Proof.   The proof will be carried out only for the case in parentheses.

1. Let L be a normed positive linear functional on $C^{\mathrm{b}}(E)$, and let

$$\lambda_L(A) := \inf\{L(f) : f \in C^{\mathrm{b}}(E), \ f \geqslant \mathbb{1}_A\}$$

for every $A \in \mathcal{A}(E)$. $\lambda_L : \mathcal{A}(E) \to [0, 1]$ is a smooth probability content in the sense of the following four properties

(a) $\lambda_L(\emptyset) = 0$, $\lambda_L(E) = 1$.

(b) $\lambda_L(A_1) \leqslant \lambda_L(A_2)$ for all $A_1, A_2 \in \mathcal{A}(E)$ with $A_1 \subset A_2$.

(c) $\lambda_L(A_1 \cup A_2) \leqslant \lambda_L(A_1) + \lambda_L(A_2)$ for all $A_1, A_2 \in \mathcal{A}(E)$, where equality holds whenever $A_1 \cap A_2 = \emptyset$.

(d) For all $A \in \mathcal{A}(E)$

$$\lambda_L(A) = \inf\{\lambda_L(O^-) : O \in \mathcal{O}(E), \ O \supset A\}.$$

2. Now, λ_L can be uniquely extended to a regular probability content $\mu_L : \mathfrak{A}(\mathcal{O}(E)) \to [0, 1]$, and it turns out that

$$L(f) = \int f \, \mathrm{d}\mu_L$$

for all $f \in C^{\mathrm{b}}(E)$.

    In order to verify this identity we pick $f \in C^{\mathrm{b}}(E)$ with $0 \leqslant f \leqslant 1$ and introduce the sets

$$G_i := \left\{ x \in E : f(x) > \frac{i}{n} \right\} \in \mathcal{O}(E)$$

for all $i = 0, 1, \ldots, n$, $n \geqslant 1$. Clearly, $G_0 \supset G_1 \supset \cdots \supset G_n = \emptyset$. Now we define functions $\alpha_i \in C\big([0, 1]\big)$ by

$$\alpha_i := \begin{cases} \equiv 0 & \text{on } \left[0, \dfrac{i-1}{n}\right] \\[2mm] \text{linear} & \text{on } \left[\dfrac{i-1}{n}, \dfrac{i}{n}\right] \\[2mm] \equiv 1 & \text{on } \left[\dfrac{i}{n}, 1\right] \end{cases}$$

and functions f_i on E by

$$f_i := \alpha_i \circ f$$

for $i = 0, 1, \ldots, n$, $n \geqslant 1$. Then

$$\frac{1}{n} \sum_{i=1}^{n} \alpha_i(t) = t$$

for all $t \in [0, 1]$, hence

$$\frac{1}{n} \sum_{i=1}^{n} f_i = f \qquad \text{and} \qquad \frac{1}{n} \sum_{i=1}^{n} L(f_i) = L(f).$$

Since $f_i \geqslant \mathbb{1}_{G_i}$, and for any $A \in \mathcal{A}(E)$, $A \subset G_i$, $\mathbb{1}_{G_i} \geqslant \mathbb{1}_A$ we obtain that $f_i \geqslant \mathbb{1}_A$ and hence that

$$L(f_i) \geqslant \lambda_L(A) = \mu_L(A).$$

From the regularity of μ_L we infer that

$$L(f_i) \geqslant \mu_L(G_i)$$

and thus

$$
\begin{aligned}
L(f) &\geqslant \frac{1}{n} \sum_{i=1}^{n} \mu_L(G_i) = \sum_{i=1}^{n} \left(\frac{i}{n} - \frac{i-1}{n} \right) \mu_L(G_i) \\
&= \sum_{i=1}^{n-1} \frac{i}{n} \left(\mu_L(G_i) - \mu_L(G_{i+1}) \right) \\
&= \left(\sum_{i=1}^{n-1} \frac{i+1}{n} \mu_L(G_i \setminus G_{i+1}) \right) - \frac{1}{n} \mu_L(G_1) \\
&\geqslant \left(\sum_{i=1}^{n-1} \int_{G_i \setminus G_{i+1}} f \, \mathrm{d}\mu_L \right) - \frac{1}{n} \mu_L(G_1) \\
&= \int_{G_1} f \, \mathrm{d}\mu_L - \frac{1}{n} \mu_L(G_1) \geqslant \int_{E} f \, \mathrm{d}\mu_L - \frac{1}{n}.
\end{aligned}
$$

For $n \to \infty$ we obtain that

$$L(f) \geqslant \int f \, \mathrm{d}\mu_L$$

whenever $f \in C^{\mathrm{b}}(E)$ with $0 \leqslant f \leqslant 1$.

But since $f \in C_+(E)$ there exists a constant $c > 0$ such that $0 \leqslant cf \leqslant 1$, hence

$$L(f) = \frac{1}{c}L(cf) \geqslant \frac{1}{c}\int cf \, \mathrm{d}\mu_L = \int f \, \mathrm{d}\mu_L \,.$$

Moreover, if $f \in C^\mathrm{b}(E)$, there exists a constant $c_1 > 0$ satisfying $f + c_1 \geqslant 0$, hence

$$L(f) = L(f + c_1) - c_1 \geqslant \int (f + c_1) \, \mathrm{d}\mu_L - c_1 = \int f \, \mathrm{d}\mu_L \,.$$

Thus we have

$$L(f) \geqslant \int f \, \mathrm{d}\mu_L$$

for all $f \in C^\mathrm{b}(E)$. Replacing f by $-f$ yields the assertion.

3. The injectivity of the correspondence $\mu \mapsto L_\mu$ can be seen as follows: Let μ, ν be regular probability contents of E satisfying

$$\int f \, \mathrm{d}\mu = \int f \, \mathrm{d}\nu$$

for all $f \in C^\mathrm{b}(E)$, and let $A \in \mathcal{A}(E)$. There exist decreasing sequences $(G_n)_{n \geqslant 1}$ and $(H_n)_{n \geqslant 1}$ in $\mathcal{O}(E)$ with $G_n \supset A$ and $H_n \supset A$ for all $n \geqslant 1$ such that

$$\lim_{n \to \infty} \mu(G_n) = \mu(A)$$

and

$$\lim_{n \to \infty} \nu(H_n) = \nu(A) \,.$$

But then $V_n := G_n \cap H_n \downarrow A$ and

$$\lim_{n \to \infty} \mu(V_n) = \mu(A)$$

as well as

$$\lim_{n \to \infty} \nu(V_n) = \nu(A) \,.$$

Choosing for every $n \geqslant 1$ a function $f_n \in C^\mathrm{b}(E)$ with the properties $0 \leqslant f_n \leqslant 1$, $f_n(A) = \{1\}$ and $f_n(\complement V_n) = \{0\}$ (the existence of which follows from $A \cap \complement V_n = \emptyset$ for all $n \geqslant 1$) we obtain

$$\int f_n \, \mathrm{d}\mu = \int_A f_n \, \mathrm{d}\mu + \int_{V_n \setminus A} f_n \, \mathrm{d}\mu = \mu(A) + \int_{V_n \setminus A} f_n \, \mathrm{d}\mu$$

and

$$\int\limits_{V_n \setminus A} f_n \, \mathrm{d}\mu \leqslant \mu(V_n \setminus A) = \mu(V_n) - \mu(A) \,,$$

hence

$$\lim_{n \to \infty} \int\limits_{V_n \setminus A} f_n \, \mathrm{d}\mu = 0 \,.$$

Therefore

$$\lim_{n \to \infty} \int f_n \, \mathrm{d}\mu = \mu(A)$$

and

$$\lim_{n \to \infty} \int f_n \, \mathrm{d}\nu = \nu(A) \,,$$

thus

$$\mu(A) = \nu(A) \quad \text{for all} \quad A \in \mathcal{A}(E)$$

and by the regularity of μ, ν also

$$\mu(B) = \nu(B) \quad \text{for all} \quad B \in \mathfrak{A}(\mathcal{A}(E)) = \mathfrak{A}(\mathcal{O}(E))$$

which implies that $\mu = \nu$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

At a later stage we will apply the following consequences of the theorem.

Corollary 1.2.2 *If for measures $\mu, \nu \in M^{\mathrm{b}}(E)$*

$$\int f \, \mathrm{d}\mu = \int f \, \mathrm{d}\nu$$

holds whenever $f \in C^{\mathrm{b}}(E)$, then $\mu = \nu$ (on $\mathfrak{B}(E)$).

Corollary 1.2.3 *Let (E, d) be a compact metric space. There is a one-to-one correspondence*

$$\mu \leftrightarrow L_\mu$$

between the set $M^{\mathrm{b}}(E)$ and the set $L^1_+(C(E))$ of positive normed linear functionals on $C(E))$ given by

$$L_\mu(f) = \int f \, \mathrm{d}\mu$$

for all $f \in C(E) = C^{\mathrm{b}}(E)$.

Proof.   The proof follows directly from Theorem 1.2.1 by applying the fact that for compact E every regular finite content on E is in fact σ-additive and hence uniquely extendable to a measure in $M^{\mathrm{b}}(E)$. For the latter property see Theorem 1.3.1.   □

We proceed to introducing a topology in $M^{\mathrm{b}}(E)$.

Definition 1.2.4 Given $\mu \in M^{\mathrm{b}}(E)$, $n \geqslant 1$, $f_1, f_2, \ldots, f_n \in C^{\mathrm{b}}(E)$ and $\varepsilon > 0$, define

$$V(\mu; f_1, f_2, \ldots, f_n; \varepsilon)$$
$$:= \left\{ \nu \in M^{\mathrm{b}}(E) : \left| \int f_i \, \mathrm{d}\mu - \int f_i \, \mathrm{d}\nu \right| < \varepsilon \text{ for all } i = 1, 2, \ldots, n \right\}.$$

The **weak topology** τ_{w} on $M^{\mathrm{b}}(E)$ is the uniquely determined topology for which

$$\left\{ V(\mu; f_1, f_2, \ldots, f_n; \varepsilon) : n \geqslant 1, \ f_1, f_2, \ldots, f_n \in C^{\mathrm{b}}(E), \ \varepsilon > 0 \right\}$$

is a neighbourhood system of μ for each $\mu \in M^{\mathrm{b}}(E)$.

Remark 1.2.5

(a) The weak topology on $M^{\mathrm{b}}(E)$ is Hausdorff due to Corollary 1.2.2.
(b) A net $(\mu_\iota)_{\iota \in I}$ in $M^{\mathrm{b}}(E)$ **converges weakly** (τ_{w}) to $\mu \in M^{\mathrm{b}}(E)$ whenever

$$\lim_{\iota \in I} \int f \, \mathrm{d}\mu_\iota = \int f \, \mathrm{d}\mu$$

for all $f \in C^{\mathrm{b}}(E)$; we write $\tau_{\mathrm{w}}\text{-}\lim_\iota \mu_\iota = \mu$.
(c) In the functional-analytic context of Appendix B.10 one introduces for the dual pair $(C^{\mathrm{b}}(E)', C^{\mathrm{b}}(E))$ of topological vector spaces the weak topology on $C^{\mathrm{b}}(E)'$. If E is compact, then Corollary 1.2.3 yields the homeomorphism

$$C^{\mathrm{b}}(E)'_+ \cong M^{\mathrm{b}}(E)$$

and consequently the coincidence of the weak topology restricted to $C^{\mathrm{b}}(E)'_+$ with the weak topology τ_{w} on $M^{\mathrm{b}}(E)$.

Definition 1.2.6 Let $\mu \in M^{\mathrm{b}}(E)$. A set $B \in \mathfrak{B}(E)$ is called a μ-**continuity** (μ-null boundary) set if $\mu(\partial B) = 0$ where $\partial B := B^- \setminus B^0$ $(\in \mathcal{A}(E))$.

Theorem 1.2.7 (Portemanteau theorem)
Let $(\mu_\iota)_{\iota \in I}$ be a net in $M^b(E)$ and $\mu \in M^b(E)$. The following statements are equivalent:

(i) $\tau_w\text{-}\lim_\iota \mu_\iota = \mu$.
(ii) $\lim_{\iota \in I} \mu_\iota(E) = \mu(E)$ *and* $\limsup_{\iota \in I} \mu_\iota(A) \leqslant \mu(A)$ *for all* $A \in \mathcal{A}(E)$.
(iii) $\lim_{\iota \in I} \mu_\iota(E) = \mu(E)$ *and* $\liminf_{\iota \in I} \mu_\iota(O) \geqslant \mu(O)$ *for all* $O \in \mathcal{O}(E)$.
(iv) $\lim_{\iota \in I} \mu_\iota(B) = \mu(B)$ *for all μ-continuity sets B.*

Proof. (i) $\Rightarrow$ (ii). As $\mathbb{1}_E \in C^b(E)$ we have $\lim_{\iota \in I} \mu_\iota(E) = \mu(E)$. Now consider $A \in \mathcal{A}(E)$. Then, as $A^{1/n} \downarrow A$ as $n \to \infty$, to each $\varepsilon > 0$ we can find $n \in \mathbb{N}$ with $\mu(A^{1/n}) - \mu(A) < \varepsilon$. Choose $f \in C^b(E)$ with $0 \leqslant f \leqslant 1$, $f(A) = \{1\}$ and $f(\complement A^{1/n}) = \{0\}$. Then

$$\limsup_{\iota \in I} \mu_\iota(A) \leqslant \limsup_{\iota \in I} \int f \, d\mu_\iota \leqslant \mu(A^{1/n}) < \mu(A) + \varepsilon$$

and hence

$$\limsup_{\iota \in I} \mu_\iota(A) \leqslant \mu(A).$$

(ii) $\Leftrightarrow$ (iii). This follows by considering complements.
(ii), (iii) $\Rightarrow$ (iv). Let B be a μ-continuity set. Then

$$\limsup_{\iota \in I} \mu_\iota(B) \leqslant \limsup_{\iota \in I} \mu_\iota(B^-) \leqslant \mu(B^-)$$
$$= \mu(B^0) \leqslant \liminf_{\iota \in I} \mu_\iota(B^0) \leqslant \liminf_{\iota \in I} \mu_\iota(B)$$

and this yields the result.

(iv) $\Rightarrow$ (i). Let $f \in C^b(E)$. Since $f(\mu)$ contains at most countably many atoms, to each $\varepsilon > 0$ there exists a strictly increasing sequence $(t_i)_{i=0,1,\ldots,k}$ in $\mathbb{R}$ with $f(\mu)(\{t_i\}) = 0$ for all $i = 0, 1, \ldots, k$, $t_i - t_{i-1} \leqslant \varepsilon$ for all $i = 1, 2, \ldots, k$, and $f(E) \subset [t_0, t_k[$. For each $i = 1, 2, \ldots, k$ put $B_i := f^{-1}([t_{i-1}, t_i[)$. Then $B_i \in \mathfrak{B}(E)$ and, since

$$\partial B_i \subset f^{-1}(\partial[t_{i-1}, t_i[) = f^{-1}(\{t_{i-1}, t_i\}),$$

we see that B_i is a μ-continuity set. We now define

$$g := \sum_{i=1}^k t_{i-1} \mathbb{1}_{B_i} \qquad \text{and} \qquad h := \sum_{i=1}^k t_i \mathbb{1}_{B_i}.$$

Then

$$g \leqslant f \leqslant g + \varepsilon \qquad \text{and} \qquad h - \varepsilon \leqslant f \leqslant h$$

and

$$\limsup_{\iota \in I} \int f \, \mathrm{d}\mu_\iota \leqslant \limsup_{\iota \in I} \int g \, \mathrm{d}\mu_\iota + \varepsilon \mu(E)$$

$$= \int g \, \mathrm{d}\mu + \varepsilon \mu(E) \leqslant \int f \, \mathrm{d}\mu + \varepsilon \mu(E)$$

and

$$\liminf_{\iota \in I} \int f \, \mathrm{d}\mu_\iota \geqslant \liminf_{\iota \in I} \int h \, \mathrm{d}\mu_\iota - \varepsilon \mu(E)$$

$$= \int h \, \mathrm{d}\mu - \varepsilon \mu(E) \geqslant \int f \, \mathrm{d}\mu - \varepsilon \mu(E)$$

as

$$\int g \, \mathrm{d}\nu = \sum_{i=1}^{k} t_{i-1} \nu(B_i) \qquad \text{and} \qquad \int h \, \mathrm{d}\nu = \sum_{i=1}^{k} t_i \nu(B_i)$$

for all $\nu \in M^{\mathrm{b}}(E)$, and E is a μ-continuity set as $\partial E = \emptyset$. It now follows that

$$\limsup_{\iota \in I} \int f \, \mathrm{d}\mu_\iota \leqslant \int f \, \mathrm{d}\mu \leqslant \liminf_{\iota \in I} \int f \, \mathrm{d}\mu_\iota$$

and this gives the desired equality. $\qquad\qquad\square$

Corollary 1.2.8 *Let* $\mu \in M^{\mathrm{b}}(E)$. *Then each of the following sets is a* τ_{w}-*neighbourhood basis of* μ.

(a) $\{\nu \in M^{\mathrm{b}}(E) : |\nu(E) - \mu(E)| < \varepsilon \text{ and } \nu(A_i) < \mu(A_i) + \varepsilon \text{ for all } i = 1, 2, \ldots, n\}$, *where* $A_1, A_2, \ldots, A_n \in \mathcal{A}(E)$, $n \in \mathbb{N}$ *and* $\varepsilon > 0$.
(b) $\{\nu \in M^{\mathrm{b}}(E) : |\nu(E) - \mu(E)| < \varepsilon \text{ and } \nu(O_i) > \mu(O_i) - \varepsilon \text{ for all } i = 1, 2, \ldots, n\}$, *where* $O_1, O_2, \ldots, O_n \in \mathcal{O}(E)$, $n \in \mathbb{N}$ *and* $\varepsilon > 0$.
(c) $\{\nu \in M^{\mathrm{b}}(E) : |\nu(B_i) - \mu(B_i)| < \varepsilon \text{ for all } i = 1, 2, \ldots, n\}$, *where* $B_1, B_2, \ldots, B_n \in \mathfrak{B}(E)$ *are* μ-*continuity sets with* $n \in \mathbb{N}$ *and* $\varepsilon > 0$.

Theorem 1.2.9 *Let* $(\mu_\iota)_{\iota \in I}$ *be a net in* $M^{\mathrm{b}}(E)$ *with* $\tau_{\mathrm{w}}\text{-}\lim_\iota \mu_\iota = \mu \in M^{\mathrm{b}}(E)$. *Furthermore let* f *be a bounded Borel-measurable real-valued function on* E. *If the set* D_f *of discontinuity points of* f *is a* μ-*null set, then*

$$\lim_{\iota \in I} \int f \, \mathrm{d}\mu_\iota = \int f \, \mathrm{d}\mu \, .$$

Proof. Let $A \in \mathcal{A}(\mathbb{R})$. Since $f^{-1}(A)^- \subset D_f \cup f^{-1}(A)$ we can apply Theorem 1.2.7 to obtain

$$\limsup_{\iota \in I} f(\mu_\iota)(A) = \limsup_{\iota \in I} \mu_\iota(f^{-1}(A)) \leqslant \limsup_{\iota \in I} \mu_\iota(f^{-1}(A)^-)$$

$$\leqslant \mu(f^{-1}(A)^-) \leqslant \mu(D_f \cup f^{-1}(A)) = \mu(f^{-1}(A)) = f(\mu)(A) \,.$$

In addition

$$\lim_{\iota \in I} f(\mu_\iota)(\mathbb{R}) = \lim_{\iota \in I} \mu_\iota(E) = \mu(E) = f(\mu)(\mathbb{R}) \,.$$

A second application of Theorem 1.2.7 gives $\tau_{\mathrm{w}}{-}\lim_\iota f(\mu_\iota) = f(\mu)$. Now consider $\varphi \in C^{\mathrm{b}}(\mathbb{R})$ such that $\mathrm{Res}_B \varphi = \mathrm{id}_B$, where B is any bounded interval containing the bounded set $f(B)$. Since $\varphi \circ f = f$ we have

$$\lim_{\iota \in I} \int f \, \mathrm{d}\mu_\iota = \lim_{\iota \in I} \int \varphi \, \mathrm{d}f(\mu_\iota) = \int \varphi \, \mathrm{d}f(\mu) = \int f \, \mathrm{d}\mu \,. \qquad \square$$

Corollary 1.2.10 *Let* $(\mu_\iota)_{\iota \in I}$ *be a net in* $M^{\mathrm{b}}(E)$ *satisfying* $\tau_{\mathrm{w}}{-}\lim_\iota \mu_\iota = \mu \in M^{\mathrm{b}}(E)$, *and let* $B \in \mathfrak{B}(E)$ *be a μ-continuity set. Then for the corresponding measures induced on B we have* $\tau_{\mathrm{w}}{-}\lim_\iota (\mu_\iota)_B = \mu_B$.

Proof. Let $f \in C^{\mathrm{b}}(E)$. Then

$$\int f \, \mathrm{d}\nu_B = \int f \mathbb{1}_B \, \mathrm{d}\nu$$

for all $\nu \in M^{\mathrm{b}}(E)$. From $D_{f\mathbb{1}_B} \subset \partial B$ we see that $D_{f\mathbb{1}_B}$ is a μ-null set. In addition $f\mathbb{1}_B$ is bounded and Borel measurable. Referring to Theorem 1.2.9 it follows that

$$\lim_{\iota \in I} \int f \, \mathrm{d}(\mu_\iota)_B = \lim_{\iota \in I} \int f \mathbb{1}_B \, \mathrm{d}\mu_\iota = \int f \, \mathrm{d}\mu_B \,. \qquad \square$$

Theorem 1.2.11 *The set* $D(E) := \{\varepsilon_x : x \in E\}$ *of Dirac measures on E is τ_{w}-closed in $M^{\mathrm{b}}(E)$, and $x \mapsto \varepsilon_x$ is a homeomorphism of E onto $D(E)$.*

Proof. Let $(x_\iota)_{\iota \in I}$ be a net in E such that $\lim_{\iota \in I} x_\iota = x \in E$. Then

$$\lim_{\iota \in I} \int f \, \mathrm{d}\varepsilon_{x_\iota} = \lim_{\iota \in I} f(x_\iota) = f(x) = \int f \, \mathrm{d}\varepsilon_x$$

for all $f \in C^{\mathrm{b}}(E)$, and thus $\tau_{\mathrm{w}}{-}\lim_\iota \varepsilon_{x_\iota} = \varepsilon_x$ which shows that $x \mapsto \varepsilon_x$ is a continuous mapping $E \to D(E)$. In addition $x \mapsto \varepsilon_x$ is injective, which is easily seen by simply choosing $B \in \mathfrak{B}(E)$ with $x \in B$ and $y \notin B$, and indeed $B = \{x\}$ will suffice.

Now suppose that $\tau_{\mathrm{w}}-\lim_\iota \varepsilon_{x_\iota} = \mu \in M^{\mathrm{b}}(E)$. From $\lim_{\iota \in I} \varepsilon_{x_\iota}(E) = \mu(E)$ we see that $\mu(E) = 1$, and hence $\mathrm{supp}(\mu) \neq \emptyset$. Choose $x \in \mathrm{supp}(\mu)$ and an open neighbourhood U of x. It follows from the properties of $\mathrm{supp}(\mu)$ and Theorem 1.2.7 that

$$\liminf_{\iota \in I} \varepsilon_{x_\iota}(U) \geqslant \mu(U) > 0$$

so that there exists $\iota_U \in I$ with $\varepsilon_{x_{\iota_U}}(U) > 0$, and hence $x_\iota \in U$ for all $\iota > \iota_U$. This shows that $\lim_{\iota \in I} x_\iota = x$, and in particular that $\varepsilon_x \mapsto x$ is continuous. It follows that $\mu = \varepsilon_x$, since τ_{w} is Hausdorff, and therefore $D(E)$ is τ_{w}-closed. $\qquad\square$

Theorem 1.2.12 *Let E, F be metric spaces, and $\varphi : E \to F$ a continuous mapping. Then $\mu \mapsto \varphi(\mu)$ is a τ_{w}-continuous mapping from $M^{\mathrm{b}}(E)$ into $M^{\mathrm{b}}(F)$.*

Proof. According to Theorem 1.1.7 we see that $\varphi(\mu) \in M^{\mathrm{b}}(F)$ for all $\mu \in M^{\mathrm{b}}(E)$. Let $(\mu_\iota)_{\iota \in I}$ be a net in $M^{\mathrm{b}}(E)$ with $\tau_{\mathrm{w}}-\lim_\iota \mu_\iota = \mu \in M^{\mathrm{b}}(E)$. For each $f \in C^{\mathrm{b}}(F)$ we have $f \circ \varphi \in C^{\mathrm{b}}(E)$, and this implies that

$$\lim_{\iota \in I} \int f \,\mathrm{d}\varphi(\mu_\iota) = \lim_{\iota \in I} \int f \circ \varphi \,\mathrm{d}\mu_\iota = \int f \circ \varphi \,\mathrm{d}\mu = \int f \,\mathrm{d}\varphi(\mu) . \qquad\square$$

In the following we will show that we can restrict the study of weak convergence to bounded sequences. For this purpose we consider the metrizability of the space $M^{\mathrm{b}}(E)$ for an arbitrary metric space (E, d).

Lemma 1.2.13 *The mapping $\rho : M^{\mathrm{b}}(E) \times M^{\mathrm{b}}(E) \to \mathbb{R}_+$ given by*

$$\varrho(\mu, \nu) = \inf\big\{\varepsilon > 0 : \mu(B) \leqslant \nu(B^\varepsilon) + \varepsilon \ \text{and} \ \nu(B) \leqslant \mu(B^\varepsilon) + \varepsilon$$
$$\text{for all } B \in \mathfrak{B}(E)\big\}$$

*for all $\mu, \nu \in M^{\mathrm{b}}(E)$ is a metric on $M^{\mathrm{b}}(E)$, called the **Prokhorov metric**. (It should be noted that the existence of numbers $\varepsilon > 0$ satisfying the above is guaranteed by the boundedness of μ, ν.)*

Proof. It is clear that $\rho(\mu, \nu) \geqslant 0$, $\rho(\mu, \nu) = \rho(\nu, \mu)$ and $\rho(\mu, \mu) = 0$ for all $\mu, \nu \in M^{\mathrm{b}}(E)$. Now suppose $\rho(\mu, \nu) = 0$. For each $A \in \mathcal{A}(E)$ we know that $A^{1/n} \downarrow A$. Then $\mu(A) = \nu(A)$ and hence by Theorem 1.1.2 (i), $\mu = \nu$.

Finally, to prove that ρ satisfies the triangle inequality, consider $\lambda, \mu, \nu \in M^{\mathrm{b}}(E)$ and $\alpha, \beta > 0$ with $\rho(\lambda, \mu) < \alpha$ and $\rho(\mu, \nu) < \beta$. By definition of $\rho(\mu, \nu)$ we have

$$\mu(B) \leqslant \nu(B^\beta) + \beta$$

for all $B \in \mathfrak{B}(E)$. From

$$(B^{\alpha})^{\beta} \cup (B^{\beta})^{\alpha} \subset B^{\alpha+\beta}$$

it follows that

$$\lambda(B) \leqslant \mu(B^{\alpha}) + \alpha \leqslant \nu((B^{\alpha})^{\beta}) + \beta + \alpha \leqslant \nu(B^{\alpha+\beta}) + (\alpha + \beta)$$

and correspondingly

$$\nu(B) \leqslant \lambda(B^{\alpha+\beta}) + (\alpha + \beta).$$

Thus $\rho(\lambda, \nu) \leqslant \alpha + \beta$. Now since $\alpha > \rho(\lambda, \mu)$ and $\beta > \rho(\mu, \nu)$ were chosen arbitrarily it follows that

$$\rho(\lambda, \nu) \leqslant \rho(\lambda, \mu) + \rho(\mu, \nu).$$

$\square$

Theorem 1.2.14 *The Prokhorov metric induces the weak topology* τ_{w} *on* $M^{\mathrm{b}}(E)$.

Proof. In each τ_{w}-neighbourhood U of $\mu \in M^{\mathrm{b}}(E)$ there is an open ρ-ball centred in μ. Without loss of generality we may assume U to be chosen as in Corollary 1.2.8 (a). Now since μ is continuous from above and $A_j^{\delta} \downarrow A_j$ as $\delta \downarrow 0$ there exists $\delta \in\,]0, \varepsilon/2[$ such that

$$\mu(A_j^{\delta}) < \mu(A_j) + \frac{\varepsilon}{2} \quad \text{for all} \quad j = 1, 2, \dots, n.$$

Choose $\nu \in M^{\mathrm{b}}(E)$ satisfying $\rho(\mu, \nu) < \delta$. Since $E^{\delta} = E$ it follows that

$$|\nu(E) - \mu(E)| \leqslant \delta < \varepsilon.$$

Furthermore

$$\nu(A_j) \leqslant \mu(A_j^{\delta}) + \delta < \mu(A_j) + \frac{\varepsilon}{2} + \delta < \mu(A_j) + \varepsilon$$

for all $j = 1, 2, \dots, n$, and this implies that $\nu \in U$.

For the reverse direction it is to be shown that each open ρ-ball centred on μ with radius $\varepsilon > 0$ contains a τ_{w}-neighbourhood U. Let $\delta \in\,]0, \varepsilon/4[$. Since μ is tight, by Theorem 1.1.4 there exists a σ-compact set $G \subset E$ with $\mu(G) = \mu(E)$. To each $x \in G$ there corresponds $\delta(x) \in\,]0, \delta/2[$ with $\mu(\partial B(x, \delta(x))) = 0$. Note that

$$\partial B(x, \eta) \subset \{y \in E : d(x, y) = \eta\}.$$

For each $n \in \mathbb{N}$ there exist finitely many $\eta > 0$ such that $\mu(\partial B(x, \eta)) \geqslant 1/n$. Thus there exist only countably many η with $\mu(\partial B(x, \eta)) \neq 0$.

Since $G \subset \bigcup_{x \in G} B(x, \delta(x))$, the σ-compactness of G yields the existence of a sequence $(x_n)_{n \geqslant 1}$ with

$$G \subset \bigcup_{n \geqslant 1} B(x_n, \delta(x_n)).$$

For each $n \in \mathbb{N}$ put $G_n := B(x_n, \delta(x_n))$. Then $(G_n)_{n \geqslant 1}$ is a sequence of μ-continuity sets with diameter less than δ, and

$$G \subset \bigcup_{n \geqslant 1} G_n.$$

Hence there exists $k \in \mathbb{N}$ such that

$$\mu \left(\bigcup_{n=1}^{k} G_n \right) > \mu(E) - \delta.$$

Put

$$\mathfrak{C} := \left\{ \bigcup_{j \in J} G_j : J \subset \{1, 2, \ldots, k\} \right\}.$$

Now $\mathfrak{C}$ is a finite system of μ-continuity sets since

$$\partial \left(\bigcup_{j \in J} G_j \right) \subset \bigcup_{j \in J} G_j^- \setminus \bigcup_{j \in J} G_j \subset \bigcup_{j \in J} \partial G_j.$$

It follows from Corollary 1.2.8 that

$$U := \{ \nu \in M^{\mathrm{b}}(E) : |\nu(E) - \mu(E)| < \delta, |\nu(C) - \mu(C)| < \delta \text{ for all } C \in \mathfrak{C} \}$$

is a τ_{w}-neighbourhood of μ.

We wish to show that for each $\nu \in U$ we have $\rho(\mu, \nu) < \varepsilon$. Put $C_0 := \bigcup_{n=1}^{k} G_n$. Since $C_0 \in \mathfrak{C}$ we have

$$\nu(C_0) > \mu(C_0) - \delta > \mu(E) - 2\delta > \nu(E) - 3\delta.$$

Now choose $B \in \mathfrak{B}(E)$. Then

$$C := \bigcup_{\{j \leqslant k \,:\, G_j \cap B \neq \emptyset\}} G_j \in \mathfrak{C},$$

$C \subset B^\delta$ (as diam $G_j < \delta$) and $B \subset C \cup \complement C_0$ (where $C \cap \complement C_0 = \emptyset$). Then

$$\mu(B) \leqslant \mu(C) + \mu(\complement C_0) < \nu(C) + \delta + \delta \leqslant \nu(B^\delta) + 2\delta < \nu(B^{4\delta}) + 4\delta$$

and analogously

$$\nu(B) \leqslant \nu(C) + \nu(\complement C_0) < \mu(C) + \delta + 3\delta \leqslant \mu(B^{4\delta}) + 4\delta.$$

Thus

$$\rho(\mu, \nu) \leqslant 4\delta < \varepsilon. \qquad \qquad \square$$

Application 1.2.15 *Let $(X_n)_{n \geqslant 0}$ be a sequence of E-valued random variables on a probability space $(\Omega, \mathfrak{A}, \mathsf{P})$ with distributions $\mathsf{P}_{X_n} \in M^1(E)$ for $n \geqslant 0$. Then*

$$X_n \to X_0 \quad \mathsf{P}\text{-stochastically}$$

implies

$$\mathsf{P}_{X_n} \to \mathsf{P}_{X_0} \quad \text{weakly as } n \to \infty.$$

In fact, given $\varepsilon > 0$ there exists $n_\varepsilon \geqslant 1$ such that

$$\mathsf{P}([d(X_n, X_0) \geqslant \varepsilon]) < \varepsilon \quad \text{for all} \quad n \geqslant n_\varepsilon.$$

Let $B \in \mathfrak{B}(E)$. Then

$$[X_n \in B] \subset [d(X_n, X_0) \geqslant \varepsilon] \cup \big([d(X_n, X_0) < \varepsilon] \cap [X_0 \in B^\varepsilon]\big)$$
$$\subset [d(X_n, X_0) \geqslant \varepsilon] \cup [X_0 \in B^\varepsilon]$$

and analogously,

$$[X_0 \in B] \subset [d(X_n, X_0) \geqslant \varepsilon] \cup [X_n \in B^\varepsilon].$$

For $n \geqslant n_\varepsilon$ follows

$$\mathsf{P}_{X_n}(B) \leqslant \varepsilon + \mathsf{P}_{X_0}(B^\varepsilon)$$

as well as

$$\mathsf{P}_{X_0}(B) \leqslant \varepsilon + \mathsf{P}_{X_n}(B^\varepsilon)$$

whenever $B \in \mathfrak{B}(E)$. But this implies

$$\rho(\mathsf{P}_{X_n}, \mathsf{P}_{X_0}) < \varepsilon,$$

thus Theorem 1.2.14 yields the assertion.

1.3   The Prokhorov theorem

As before (E, d) is a metric space, and the space $M^{\mathrm{b}}(E)$ is given the topology τ_{w}, or equivalently the Prokhorov metric ρ. Hence $(M^{\mathrm{b}}(E), \rho)$ is a metric space.

A finite content μ on E is called **inner-regular** if

$$\mu(B) = \sup\{\mu(K) : K \in \mathcal{K}(E), \ K \subset B\}$$

for all $B \in \mathfrak{B}(E)$.

Theorem 1.3.1 *Every inner-regular content μ on E is σ-additive, so that $\mu \in M^{\mathrm{b}}(E)$.*

Proof.   All that needs to be shown is that μ is $\emptyset$-continuous. Let $(B_n)_{n \geqslant 1}$ be a sequence in $\mathfrak{B}(E)$ with $B_n \downarrow \emptyset$, and ε be given. For each $n \in \mathbb{N}$ there exists a compact set $K_n \subset B_n$ such that

$$\mu(B_n) - \mu(K_n) < \frac{1}{2^n}\,\varepsilon\,.$$

Put $L_n := \bigcap_{i=1}^{n} K_i$. Then

$$\mu(B_n) - \mu(L_n) \leqslant \sum_{i=1}^{n}(\mu(B_i) - \mu(K_i)) < \varepsilon$$

for all $n \in \mathbb{N}$, and $L_n \downarrow \emptyset$. From the finite intersection property there exists $n_0 \in \mathbb{N}$ such that $L_n = \emptyset$ for all $n \geqslant n_0$, and hence $\mu(L_n) = 0$ for all $n \geqslant n_0$. Then $\mu(B_n) < \varepsilon$ for all $n \geqslant n_0$, and we have shown that $\lim_{n \to \infty} \mu(B_n) = 0$. $\qquad\square$

Corollary 1.3.2 *Let $(\mu_n)_{n \geqslant 1}$ be an increasing sequence in $M^{\mathrm{b}}(E)$ satisfying*

$$\sup_{n \geqslant 1} \mu_n(E) < \infty\,.$$

Then $\sup_{n \geqslant 1} \mu_n \in M^{\mathrm{b}}(E)$.

Proof.   Write $\mu(B) := \sup_{n \geqslant 1} \mu_n(B)$ for all $B \in \mathfrak{B}(E)$. Clearly μ is a finite content on $\mathfrak{B}(E)$. From Theorem 1.1.2 we have that μ_n is inner-regular, and so μ is an inner-regular content, and the result follows from Theorem 1.3.1. $\qquad\square$

Theorem 1.3.3 *Let (E, d) be a compact metric space. Then for each $a > 0$*

$$M^{(a)}(E) := \{\mu \in M^{\mathrm{b}}(E) : \mu(E) \leqslant a\}$$

is τ_{w}-compact.

Proof.   According to Appendix B.11 (Alaoglu, Bourbaki), the set

$$V^{(a)} := \{\varphi \in C^{b}(E)' : \|\varphi\| \leqslant a\}$$

is weakly compact, i.e. compact with respect to the topology $\sigma(C^{b}(E)', C^{b}(E))$. Hence

$$V_{+}^{(a)} := \{\varphi \in V^{(a)} : \varphi \geqslant 0\} = \bigcap_{f \in C_{+}^{b}(E)} \{\varphi \in V^{(a)} : \varphi(f) \geqslant 0\}$$

is weakly compact. Furthermore by Corollary 1.2.3 the mapping

$$\mu \mapsto \left(f \mapsto \int f \, d\mu\right)$$

is a bijection from $M^{(a)}(E)$ onto $V_{+}^{(a)}$. From the definition of τ_{w} it follows that it is a homeomorphism, and hence that $M^{(a)}(E)$ is τ_{w}-compact. $\square$

Definition 1.3.4 A set $H \subset M^{b}(E)$ is called **uniformly tight** if

(a) $\sup\{\mu(E) : \mu \in H\} < \infty$;
(b) to each $\varepsilon > 0$ there exists a compact set $K \subset E$ such that

$$\mu(\complement K) < \varepsilon \quad \text{for all} \quad \mu \in H\,.$$

Theorem 1.3.5 *Suppose (E, d) is separable complete. For each $H \subset M^{b}(E)$ the following statements are equivalent:*

 (i) *H is uniformly tight.*
(ii) (a) $\sup\{\mu(E) : \mu \in H\} < \infty$;
      (b) *For all $\varepsilon > 0$ and $n \in \mathbb{N}$ there exist $x_1, x_2, \ldots, x_k \in E$ such that*

$$\mu(\complement B_n) < \varepsilon \quad \text{for all} \quad \mu \in H$$

   where $B_n := \bigcup_{j=1}^{k} B\left(x_j, \frac{1}{n}\right)$.

Proof.   (i) $\Rightarrow$ (ii). To each $\varepsilon > 0$ there exists a compact set $K \subset E$ such that $\mu(\complement K) < \varepsilon$ for all $\mu \in H$. Also to each $n \in \mathbb{N}$ there exist $x_1, x_2, \ldots, x_k \in E$ such that $K \subset B_n$, so that

$$\mu(\complement B_n) < \varepsilon$$

for all $\mu \in H$.

   (ii) $\Rightarrow$ (i). Let $\varepsilon > 0$. From the assumption for each $n \in \mathbb{N}$ there exists $B_n \subset E$ such that B_n is the finite union of $\frac{1}{n}$-balls and

$$\mu(\complement B_n) < \frac{1}{2^n}\,\varepsilon$$

for all $\mu \in H$. Put $L := \bigcap_{n \geqslant 1} B_n$. Then $\mu(\complement L) < \varepsilon$ for all $\mu \in H$. Now L is totally bounded, and hence so is L^-. As E is complete it follows that L^- is compact. The result now follows from the fact that

$$\mu(\complement L^-) \leqslant \mu(\complement L) < \varepsilon \quad \text{for all} \quad \mu \in H .$$

$\square$

Lemma 1.3.6 Let $A \in \mathcal{A}(E)$, $\mu_\iota \in M^{\mathrm{b}}(E)$ for all $\iota \in I$, $\mu \in M^{\mathrm{b}}(E)$, $\mu_A \in M^{\mathrm{b}}(A)$. If $\tau_{\mathrm{w}}-\lim_\iota \mu_\iota = \mu$ and $\tau_{\mathrm{w}}-\lim_\iota \operatorname{Res}_A \mu_\iota = \mu_A$ then $\mu_A \leqslant \operatorname{Res}_A \mu$.

Proof.   We can use Theorem 1.1.2 (ii) to deduce that $\operatorname{Res}_A \mu \in M^{\mathrm{b}}(A)$. Consider a continuous function $g : A \to [0,1]$.   By Tietze's extension theorem there exists a continuous function $f : E \to [0,1]$ with $\operatorname{Res}_A f = g$. Then

$$\int g \, \mathrm{d}\mu_A = \lim_{\iota \in I} \int g \, \mathrm{d}(\operatorname{Res}_A \mu_\iota) = \lim_{\iota \in I} \int_A f \, \mathrm{d}\mu_\iota$$

and

$$\int g \, \mathrm{d}(\operatorname{Res}_A \mu) = \int_A f \, \mathrm{d}\mu .$$

But in slight modification of Corollary 1.2.2 one obtains that for measures $\mu, \nu \in M^{\mathrm{b}}(E)$ satisfying

$$\int f \, \mathrm{d}\mu \leqslant \int f \, \mathrm{d}\nu \quad \text{for all} \quad f \in C^{\mathrm{b}}(E) \quad \text{with} \quad 0 \leqslant f \leqslant 1$$

$\mu \leqslant \nu$ holds. Therefore we need only prove that

$$\lim_{\iota \in I} \int_A f \, \mathrm{d}\mu_\iota \leqslant \int_A f \, \mathrm{d}\mu .$$

However this inequality follows immediately from the Portemanteau theorem 1.2.7, as clearly $\tau_{\mathrm{w}}-\lim_\iota f \cdot \mu_\iota = f \cdot \mu$. $\square$

Theorem 1.3.7 (Prokhorov)
Consider $H \subset M^{\mathrm{b}}(E)$.

(i) *If H is uniformly tight then H is τ_{w}-relatively compact.*
(ii) *If E is separable complete then any τ_{w}-relatively compact set H is uniformly tight.*

Proof.   (i). Let $(K_n)_{n \geqslant 1}$ be an increasing sequence of compact subsets of E satisfying $\mu(\complement K_n) < 1/n$ for all $\mu \in H$. Let $(\mu_k)_{k \geqslant 1}$ be a

sequence in H. We have to show that $(\mu_k)_{k \geqslant 1}$ possesses a τ_{w}-convergent subsequence. From

$$a := \sup\{\mu(K_n) : \mu \in H,\, n \in \mathbb{N}\} \leqslant \sup\{\mu(E) : \mu \in H\} < \infty$$

appealing to Theorem 1.3.3 we have that $\{\operatorname{Res}_{K_n} \mu : \mu \in H\} \,(\subset M^{(a)}(K_n))$ is τ_{w}-relatively compact in $M^{\mathrm{b}}(K_n)$. By the metrizability of $M^{\mathrm{b}}(E)$, which is the content of Theorem 1.2.14, a diagonal argument provides for each $n \in \mathbb{N}$ a measure $\nu_n' \in M^{\mathrm{b}}(K_n)$ with

$$\tau_{\mathrm{w}}-\lim_k \operatorname{Res}_{K_n} \mu_k = \nu_n' \,.$$

Put

$$\nu_n(B) := \nu_n'(B \cap K_n) \quad \text{for all} \quad B \in \mathfrak{B}(E) \,.$$

Then $\nu_n \in M^{\mathrm{b}}(E)$ for all $n \in \mathbb{N}$. Also from Lemma 1.3.6, $\nu_n' \leqslant \operatorname{Res}_{K_n} \nu_{n+1}'$ and (recall that $K_n \subset K_{n+1}$)

$$\nu_n(B) \leqslant \nu_{n+1}'(B \cap K_n) \leqslant \nu_{n+1}(B)$$

for all $B \in \mathfrak{B}(E)$ which says that $(\nu_n)_{n \geqslant 1}$ is an increasing sequence. Moreover,

$$\nu_n(E) = \nu_n'(K_n) = \lim_{k \to \infty} \mu_k(K_n) \leqslant \liminf_{k \to \infty} \mu_k(E) < \infty$$

for all $n \in \mathbb{N}$. Now appeal to Corollary 1.3.2 to obtain

$$\nu := \sup_{n \geqslant 1} \nu_n \in M^{\mathrm{b}}(E)$$

and moreover

$$\nu(E) = \sup_{n \geqslant 1} \nu_n(E) \leqslant \liminf_{k \to \infty} \mu_k(E) \,.$$

Let $A \in \mathcal{A}(E)$. Applying the Portemanteau theorem 1.2.7 we get

$$\nu(A) \geqslant \nu_n(A) = \nu_n'(A \cap K_n) \geqslant \limsup_{k \geqslant 1} \mu_k(A \cap K_n)$$

$$\geqslant \limsup_{k > 1} \mu_k(A) - \frac{1}{n}$$

for all $n \in \mathbb{N}$, where for the second inequality we have used the fact that $A \cap K_n$ is closed, and for the third that

$$\mu(A \cap K_n) \geqslant \mu(A) - \frac{1}{n}$$

for all $\mu \in H$. Hence

$$\nu(A) \geqslant \limsup_{k \geqslant 1} \mu_k(A)$$

and

$$\nu(E) = \lim_{k \to \infty} \mu_k(E) \,.$$

A further application of the Portemanteau theorem 1.2.7 gives $\tau_{\mathrm{w}}-$ $\lim_k \mu_k = \nu$. Thus every sequence in H has a τ_{w}-convergent subsequence, and this just says that H is τ_{w}-relatively compact.

(ii). Assume that H is τ_{w}-relatively compact and at the same time fails to be uniformly tight. The mapping

$$\mu \mapsto \mu(E) = \int \mathbb{1}_E \, \mathrm{d}\mu$$

is continuous, and this implies that

$$\sup\{\mu(E) : \mu \in H\} < \infty \,.$$

Let $\mathcal{F} = \{F \subset E : |F| < \infty\}$. Theorem 1.3.5 implies that there exist $\varepsilon > 0$ and $n \in \mathbb{N}$ such that to each $F \in \mathcal{F}$ there is $\mu_F \in H$ with

$$\mu_F \left(E \setminus \left(\bigcup_{x \in F} B\left(x, \frac{1}{n}\right) \right) \right) \geqslant \varepsilon \,.$$

The net $(\mu_F)_{F \in \mathcal{F}}$ contains a τ_{w}-convergent subnet $(\mu_{F_\iota})_{\iota \in I}$ with limit μ say. For each $\iota \in I$ define

$$B_\iota := \bigcup_{x \in F_\iota} B\left(x, \frac{1}{n}\right) \,.$$

Now $(\complement B_\iota)_{\iota \in I}$ is a downward filtered family in $\mathcal{A}(E)$ with $\bigcap_{\iota \in I} \complement B_\iota = \emptyset$. It follows from Corollary 1.1.3 that

$$\inf_{\iota \in I} \mu(\complement B_\iota) = 0 \,.$$

On the other hand applying the Portemanteau theorem we have

$$\mu(\complement B_\kappa) \geqslant \limsup_{\iota \in I} \mu_{F_\iota}(\complement B_\kappa) \geqslant \limsup_{\iota \in I} \mu_{F_\iota}(\complement B_\iota) \geqslant \varepsilon$$

for all $\kappa \in I$, and this is a contradiction. $\qquad\square$

1.4 Convolution of measures

In this section (E, d) will denote a separable complete metric Abelian group which means that E is an Abelian group (with binary operation denoted by addition and 0 as neutral element), (E, d) is a separable and complete metric space with distance function d, and the mapping $(x, y) \mapsto x - y$ from $E \times E$ into E is continuous. Along with (E, d), $(E \times E, d \times d)$ is also a separable complete metric Abelian group. Here the metric $d \times d$ is defined by $d \times d((x, y), (u, v)) := \max\{d(x, u), d(y, v)\}$ for all $(x, y), (u, v) \in E$. A prominent example of a separable complete metric Abelian group is a separable Banach space $(E, \| \cdot \|)$ over $\mathbb{R}$, where the distance function is given by $d(x, y) := \|x - y\|$ for all $x, y \in E$.

For separable complete metric groups E we have that each finite measure on E is tight (Theorem 1.1.6) and that the notions "uniform tightness" and "τ_{w}-relative compactness" are equivalent (Prokhorov's theorem 1.3.7).

Theorem 1.4.1 *Let (E, d) be a separable complete metric Abelian group.*

(i) $\mathfrak{B}(E \times E) = \mathfrak{B}(E) \otimes \mathfrak{B}(E)$.
(ii) *The mapping $(x, y) \mapsto m(x, y) := x + y$ from $E \times E$ into E is $(\mathfrak{B}(E) \otimes \mathfrak{B}(E), \mathfrak{B}(E))$-measurable.*

Proof. (i). $\mathcal{O}(E)$ is a generator of $\mathfrak{B}(E)$ with the exhaustion property which says that there exists a sequence $(O_n)_{n \geqslant 1}$ in $\mathcal{O}(E)$ such that $O_n \uparrow E$. Thus $\mathcal{O}(E) \times \mathcal{O}(E)$ is a generator of $\mathfrak{B}(E) \otimes \mathfrak{B}(E)$. Furthermore $\mathcal{O}(E) \times \mathcal{O}(E) \subset \mathcal{O}(E \times E)$ from which follows that $\mathfrak{B}(E) \otimes \mathfrak{B}(E) \subset \mathfrak{B}(E \times E)$. Now choose a countable dense subset D of E, so that $D \times D$ is a countable dense subset of $E \times E$. The open balls in E and $E \times E$ centred at points in D and $D \times D$ respectively and with rational radii make up countable bases $\mathcal{B}$ and $\mathcal{C}$ in $\mathcal{O}(E)$ and $\mathcal{O}(E \times E)$ respectively. Now

$$B((x, y), r) = B(x, r) \times B(y, r)$$

for the corresponding balls, and hence $\mathcal{B} \times \mathcal{B} \supset \mathcal{C}$. Now $\mathcal{B}$ and $\mathcal{C}$ are generators of $\mathfrak{B}(E)$ and $\mathfrak{B}(E \times E)$ respectively with the exhaustion property, which implies that $\mathfrak{B}(E) \otimes \mathfrak{B}(E) \supset \mathfrak{B}(E \times E)$.

(ii). The mapping m from $E \times E$ into E is continuous, and hence $(\mathfrak{B}(E \times E), \mathfrak{B}(E))$-measurable. Now apply (i). $\qquad \square$

Application 1.4.2 *Let X, Y be E-valued random variables on a probability space $(\Omega, \mathfrak{A}, \mathsf{P})$. Then by Theorem 1.4.1 (ii) the mapping*

$\omega \mapsto (X + Y)(\omega) = X(\omega) + Y(\omega)$ *from* Ω *into* E *is also an E-valued random variable on* $(\Omega, \mathfrak{A}, \mathsf{P})$, *since* $X + Y = m \circ (X, Y)$.

Definition 1.4.3 For $\mu, \nu \in M^{\mathrm{b}}(E)$ we refer to the measure $\mu * \nu := m(\mu \otimes \nu)$ on E as the **convolution** of μ and ν.

We have the following properties of the convolution mapping.

Properties 1.4.4

1.4.4.1 *For all* $f \in C^{\mathrm{b}}(E)$

$$\int f \, \mathrm{d}(\mu * \nu) = \int \left(\int f(x + y) \, \mu(\mathrm{d}x) \right) \nu(\mathrm{d}y)$$

$$= \int \left(\int f(x + y) \, \nu(\mathrm{d}y) \right) \mu(\mathrm{d}x)$$

which follows from Fubini's theorem.

1.4.4.2 *In particular, for all* $B \in \mathfrak{B}(E)$

$$(\mu * \nu)(B) = \int \mu(B - y) \, \nu(\mathrm{d}y) = \int \nu(B - x) \, \mu(\mathrm{d}x) \, .$$

1.4.4.3 *For all* $B, C \in \mathfrak{B}(E)$ *we have* $B + C \in \mathfrak{B}(E)$ *and*

$$(\mu * \nu)(B + C) \geqslant \mu(B)\nu(C)$$

The latter can easily be seen from the inequalities

$$(\mu * \nu)(B + C) = (\mu \otimes \nu)(m^{-1}(B + C)) \geqslant (\mu \otimes \nu)(B \times C) = \mu(B)\nu(C) \, .$$

1.4.4.4 *The convolution is commutative and associative, that is*

$$\mu * \nu = \nu * \mu$$

and

$$(\lambda * \mu) * \nu = \lambda * (\mu * \nu)$$

for all $\lambda, \mu, \nu \in M^{\mathrm{b}}(E)$.

The proof uses 1.4.4.1 in conjunction with the injectivity of the mapping

$$\mu \mapsto \left(f \mapsto \int f \, \mathrm{d}\mu \right)$$

from $M^{\mathrm{b}}(E)$ into $C^{\mathrm{b}}(E)'_+$.

1.4.4.5 *For each* $x \in E$ *and* $B \in \mathfrak{B}(E)$

$$(\mu * \varepsilon_x)(B) = \mu(B - x) \, .$$

1.4.4.6 *From 1.4.4.5 and 1.4.4.2 we have* $\mu * \varepsilon_0 = \mu$ *and* $\varepsilon_x * \varepsilon_y = \varepsilon_{x+y}$ *for all* $x, y \in E$.

Application 1.4.5 *Let X, Y be independent E-valued random variables on a probability space $(\Omega, \mathfrak{A}, \mathsf{P})$. Then $\mathsf{P}_{X+Y} = \mathsf{P}_X * \mathsf{P}_Y$.*

Proof. To show this we note that independence of X, Y gives $\mathsf{P}_{(X,Y)} = \mathsf{P}_X \otimes \mathsf{P}_Y$ which implies that

$$\mathsf{P}_{X+Y} = \mathsf{P}_{m \circ (X,Y)} = m(\mathsf{P}_{(X,Y)}) = m(\mathsf{P}_X \otimes \mathsf{P}_Y) = \mathsf{P}_X * \mathsf{P}_Y \,.$$
$\square$

Theorem 1.4.6 (Support formula)
For $\mu, \nu \in M^{\mathrm{b}}(E)$

$$\mathrm{supp}(\mu * \nu) = (\mathrm{supp}(\mu) + \mathrm{supp}(\nu))^- \,.$$

Proof. First we note that

$$
\begin{aligned}
(\mu * \nu)((\mathrm{supp}(\mu) + \mathrm{supp}(\nu))^-) &= (\mu \otimes \nu)(m^{-1}((\mathrm{supp}(\mu) + \mathrm{supp}(\nu))^-)) \\
&\geqslant (\mu \otimes \nu)(\mathrm{supp}(\mu) \times \mathrm{supp}(\nu)) = \mu(E)\nu(E) \\
&= (\mu \otimes \nu)(E \times E) = (\mu * \nu)(E) \,.
\end{aligned}
$$

Thus by Theorem 1.1.4 (i)

$$\mathrm{supp}(\mu * \nu) \subset (\mathrm{supp}(\mu) + \mathrm{supp}(\nu))^- \,.$$

To prove the reverse inclusion, consider $x \in \mathrm{supp}(\mu)$, $U \in \mathfrak{V}(x)$, $y \in \mathrm{supp}(\nu)$ and $V \in \mathfrak{V}(y)$. Then $U + V \in \mathfrak{V}(x+y)$. Now Property 1.4.4.3 together with Theorem 1.1.4 (iii) gives

$$(\mu * \nu)(U + V) \geqslant \mu(U)\mu(V) > 0 \,.$$

To each $W \in \mathfrak{V}(x+y)$ there exist $U \in \mathfrak{V}(x)$, $V \in \mathfrak{V}(y)$ with $U + V \subset W$. A further application of Theorem 1.1.4 (iii) gives $x+y \in \mathrm{supp}(\mu+\nu)$. Thus

$$\mathrm{supp}(\mu) + \mathrm{supp}(\nu) \subset \mathrm{supp}(\mu * \nu)$$

and this completes the proof.
$\square$

Corollary 1.4.7 *If $\mu * \nu$ is a Dirac measure then so is each of the measures μ and ν.*

Proof. We just observe that a measure is Dirac precisely when it has a single element support.
$\square$

Theorem 1.4.8 *Let $(\mu_n)_{n \geqslant 1}$, $(\nu_n)_{n \geqslant 1}$ be sequences with $\tau_{\mathrm{w}}\text{-}\lim_n \mu_n = \mu$, $\tau_{\mathrm{w}}\text{-}\lim_n \nu_n = \nu$ in $M^{\mathrm{b}}(E)$. Then $\tau_{\mathrm{w}}\text{-}\lim_n \mu_n \otimes \nu_n = \mu \otimes \nu$.*

Proof.   Appealing to Prokhorov's theorem 1.3.7 to each $\varepsilon > 0$ there exist $K \in \mathcal{K}(E)$ with

$$\mu_n(\complement K) < \varepsilon \qquad \text{and} \qquad \nu_n(\complement K) < \varepsilon$$

and $\alpha > 0$ such that

$$\mu_n(E) \leqslant \alpha \qquad \text{and} \qquad \nu_n(E) \leqslant \alpha$$

for all $n \in \mathbb{N}$. Clearly $K \times K \in \mathcal{K}(E \times E)$, and

$$
\begin{aligned}
(\mu_n \otimes \nu_n)(K \times K) = \mu_n(K)\nu_n(K) &\geqslant (\mu_n(E) - \varepsilon)(\nu_n(E) - \varepsilon) \\
&> (\mu_n \otimes \nu_n)(E \times E) - \varepsilon(\mu_n(E) + \nu_n(E)) \,.
\end{aligned}
$$

Therefore

$$(\mu_n \otimes \nu_n)(\complement(K \times K)) < 2\alpha\varepsilon$$

and

$$(\mu_n \otimes \nu_n)(E \times E) = \mu_n(E)\nu_n(E) \leqslant \alpha^2$$

for all $n \in \mathbb{N}$. Furthermore from Prokhorov's theorem 1.3.7 we see that $\{\mu_n \otimes \nu_n : n \in \mathbb{N}\}$ is τ_w-relatively compact.

Let $\lambda \in M^b(E \times E)$ be a cluster point of the sequence $(\mu_n \otimes \nu_n)_{n \geqslant 1}$, that is, there exists a subsequence $(\mu_{n_k} \otimes \nu_{n_k})_{k \geqslant 1}$ with limit λ. Now

$$
\begin{aligned}
\mathcal{E} := \{ B_1 \times B_2 \in \mathfrak{B}(E) \times \mathfrak{B}(E) : \\
B_1 \times B_2 \text{ is both } \mu \otimes \nu\text{- and a } \lambda\text{-continuity set} \}
\end{aligned}
$$

is an $\cap$-stable generator of $\mathfrak{B}(E) \otimes \mathfrak{B}(E) = \mathfrak{B}(E \times E)$. Let $B_1 \times B_2 \in \mathcal{E}$. Then

$$\partial(B_1 \times B_2) = (B_1^- \times \partial B_2) \cup (\partial B_1 \times B_2^-) \,.$$

Therefore we have the following three cases to consider.

(1) B_1 is a μ-continuity set and B_2 is a ν-continuity set.
(2) $\mu(B_1^-) = 0$.
(3) $\nu(B_2^-) = 0$.

With the help of the Portemanteau theorem 1.2.7 we have

$$
\begin{aligned}
\lambda(B_1 \times B_2) = \lim_{k \to \infty} (\mu_{n_k} \otimes \nu_{n_k})(B_1 \times B_2) &= \lim_{k \to \infty} \mu_{n_k}(B_1)\nu_{n_k}(B_2) \\
&= \mu(B_1)\nu(B_2) = (\mu \otimes \nu)(B_1 \times B_2) \,,
\end{aligned}
$$

$$\lambda(B_1 \times B_2) \leqslant \lim_{k \to \infty} \mu_{n_k}(B_1)\alpha = \mu(B_1)\alpha = 0 = (\mu \otimes \nu)(B_1 \times B_2) \,.$$

and similarly in the remaining case.

Thus

$$\lambda(B_1 \times B_2) = (\mu \otimes \nu)(B_1 \times B_2) \quad \text{for all} \quad B_1 \times B_2 \in \mathcal{E}$$

and hence $\lambda = \mu \otimes \nu$. Hence the τ_{w}-relatively compact sequence $(\mu_n \otimes \nu_n)_{n \geqslant 1}$ has a unique cluster point $\mu \otimes \nu$, and it follows that $\tau_{\mathrm{w}}-\lim_n \mu_n \otimes \nu_n = \mu \otimes \nu$. $\qquad \square$

Theorem 1.4.9 *Let (E, d) be a separable complete metric Abelian group. Then the space $(M^{\mathrm{b}}(E), \tau_{\mathrm{w}}, *)$ with the convolution $*$ defined above is a commutative metric semigroup, in the sense that*

  (i) *$(M^{\mathrm{b}}(E), \tau_{\mathrm{w}})$ is a metric space (with the Prokhorov metric inducing the topology τ_{w}),*
 (ii) *$(M^{\mathrm{b}}(E), *)$ is a commutative semigroup with neutral element ε_0, that is, $\varepsilon_0 * \mu = \mu$ for all $\mu \in M^{\mathrm{b}}(E)$, and*
(iii) *the mapping $(\mu, \nu) \mapsto \mu * \nu$ from $M^{\mathrm{b}}(E) \times M^{\mathrm{b}}(E)$ into $M^{\mathrm{b}}(E)$ is τ_{w}-continuous.*

Moreover,

 (iv) *$(M^1(E), \tau_{\mathrm{w}}, *)$ is a subsemigroup of $(M^{\mathrm{b}}(E), \tau_{\mathrm{w}}, *)$.*

Proof. In view of the assumed metrizability it suffices to consider sequences. So given sequences $(\mu_n)_{n \geqslant 1}$, $(\nu_n)_{n \geqslant 1}$ with $\tau_{\mathrm{w}}-\lim_n \mu_n = \mu$, $\tau_{\mathrm{w}}-\lim_n \nu_n = \nu$ in $M^{\mathrm{b}}(E)$ we have by Theorem 1.4.8 that $\tau_{\mathrm{w}}-\lim_n \mu_n \otimes \nu_n = \mu \otimes \nu$. Then Theorem 1.2.12 gives

$$\tau_{\mathrm{w}}-\lim_n \mu_n * \nu_n = \tau_{\mathrm{w}}-\lim_n m(\mu_n \otimes \nu_n) = m(\mu \otimes \nu) = \mu * \nu.$$

This proves the continuity of $(\mu, \nu) \mapsto \mu * \nu$. The remaining assertions follow immediately from Properties 1.4.4.2 and 1.4.4.4 of the convolution.

$\qquad \square$

Chapter 2

The Fourier Transform
in a Banach Space

2.1   Fourier transforms of probability measures

Let $(E, \|\cdot\|)$ denote a separable Banach space over $\mathbb{R}$. With the metric

$$(x, y) \mapsto \|x - y\|$$

$(E, \|\cdot\|)$ is a complete metric space. We use E' to designate the topological dual of E. For $x \in E$ and any (continuous real-valued) linear functional $a \in E'$ we put $\langle x, a \rangle := a(x)$. It is well-known that E' separates the points of E, in the sense that for every $x \in \complement\{0\}$ there exists $a \in E'$ satisfying $\langle x, a \rangle \neq 0$ (see Appendix B).

Now let $\mathcal{N}(E)$ denote the family of all closed sub (vector) spaces E of finite codimension. For every $a \in E'$ we have $\ker a \in \mathcal{N}(E)$, and therefore $\bigcap\{N : N \in \mathcal{N}(E)\} = \{0\}$. Finally, for every $N \in \mathcal{N}(E)$ let p_N denote the canonical mapping from E onto E/N.

Theorem 2.1.1 *For every $K \in \mathcal{K}(E)$ we have*

$$K = \bigcap\{p_N^{-1}(p_N(K)) : N \in \mathcal{N}(E)\}.$$

Proof.   Clearly

$$K \subset K_1 := \bigcap\{p_N^{-1}(p_N(K)) : N \in \mathcal{N}(E)\}.$$

For the reverse inclusion, let $x \in K_1$. Then for every $N \in \mathcal{N}(E)$ the set

$$K_N := p_N^{-1}(p_N(x)) \cap K$$

is compact and non-empty. Moreover, given $M, N \in \mathcal{N}(E)$ with $M \subset N$ we have $K_M \subset K_N$. Indeed, let $p_{N,M}$ denote the canonical mapping from E/M onto E/N satisfying $p_{N,M} \circ p_M = p_N$. Then for $y \in K_M$,

$$p_N(y) = p_{N,M}(p_M(y)) = p_{N,M}(p_M(x)) = p_N(x),$$

and thus $y \in K_N$.

Next we observe that $\mathcal{N}(E)$ is $\cap$-stable, and hence that $(K_N)_{N \in \mathcal{N}(E)}$ is a downward filtered family. Since each K_N is compact, $\bigcap \{K_N : N \in \mathcal{N}(E)\} \neq \emptyset$. But for $y \in \bigcap \{K_N : N \in \mathcal{N}(E)\}$ we have $p_N(y) = p_N(x)$ whenever $N \in \mathcal{N}(E)$. From $\bigcap \{N : N \in \mathcal{N}(E)\} = \{0\}$ we have $x = y \in K$ and consequently $K_1 \subset K$. $\qquad \square$

Definition 2.1.2 Let $\mu \in M^b(E)$. The mapping $\hat{\mu} : E' \to \mathbb{C}$ given by

$$\hat{\mu}(a) := \int e^{\mathrm{i}\langle x,a\rangle}\, \mu(\mathrm{d}x)$$

for all $a \in E'$ is called the **Fourier transform** of μ.

For Banach spaces E, F denote by $L(E, F)$ the set of all continuous linear mappings from E into F, and consider $T \in L(E, F)$. The adjoint $T^\top$ of T is continuous linear mapping from F' into E' given by $\langle x, T^\top b\rangle = \langle Tx, b\rangle$ whenever $x \in E$, $b \in F'$.

Lemma 2.1.3 *Let* $\mu \in M^b(E)$. *Then for any* $T \in L(E, F)$

$$T(\mu)^\wedge = \hat{\mu} \circ T^\top\,.$$

Proof. We know already that $T(\mu) \in M^b(F)$ whenever $\mu \in M^b(E)$. Then for every $b \in F'$,

$$T(\mu)^\wedge(b) = \int e^{\mathrm{i}\langle y,b\rangle}\, T(\mu)(\mathrm{d}y) = \int e^{\mathrm{i}\langle Tx,b\rangle}\, \mu(\mathrm{d}x)$$

$$= \int e^{\mathrm{i}\langle x,T^\top b\rangle}\, \mu(\mathrm{d}x) = \hat{\mu}(T^\top b)$$

$$= (\hat{\mu} \circ T^\top)(b)\,. \qquad \square$$

Theorem 2.1.4 (Uniqueness of the Fourier transform)
Let $\mu, \nu \in M^b(E)$ *with* $\hat{\mu} = \hat{\nu}$. *Then* $\mu = \nu$.

Proof. From Lemma 2.1.3 we deduce that

$$p_N(\mu)^\wedge = p_N(\nu)^\wedge$$

for every $N \in \mathcal{N}(E)$. But from Appendix B.3 we infer that $E/N \cong \mathbb{R}^p$ with $p := \dim(E/N)$ (in the sense of a topological isomorphism). Applying Property 4.2.16.3 on the uniqueness of the classical Fourier transform of bounded measures on $\mathbb{R}^p$ we obtain that $p_N(\mu) = p_N(\nu)$ for every $N \in \mathcal{N}(E)$. Then

$$(p_N^{-1}(p_N(K)))_{N \in \mathcal{N}(E)}$$

is a downward filtered family in $\mathcal{A}(E)$ satisfying (by Theorem 2.1.1) the equality

$$K = \bigcap \{p_N^{-1}(p_N(K)) : N \in \mathcal{N}(E)\}\,.$$

It follows that

$$\begin{aligned}
\mu(K) &= \inf\{\mu(p_N^{-1}(p_N(K))) : N \in \mathcal{N}(E)\} \\
&= \inf\{p_N(\mu)(p_N(K)) : N \in \mathcal{N}(E)\} \\
&= \inf\{p_N(\nu)(p_N(K)) : N \in \mathcal{N}(E)\} \\
&= \inf\{\nu(p_N^{-1}(p_N(K))) : N \in \mathcal{N}(E)\} \\
&= \nu(K)\,.
\end{aligned}$$

From Theorems 1.1.6 and 1.1.2 (ii) we now infer that $\mu = \nu$. $\qquad\square$

Corollary 2.1.5 *Every $\mu \in M^{\mathrm{b}}(E)$ is uniquely determined by the family $\{a(\mu) : a \in E'\}$ of its one-dimensional marginal distributions.*

Proof. For $x \in E$, $a \in E'$ and $t \in \mathbb{R}$ we have

$$\langle x, a^\top(t)\rangle = \langle a(x), t\rangle = ta(x) = a(tx) = \langle tx, a\rangle = \langle x, ta\rangle$$

and hence $a^\top(1) = a$. Then Lemma 2.1.3 yields $a(\mu)^\wedge(1) = \hat{\mu}(a)$, and Theorem 2.1.4 yields the assertion. $\qquad\square$

From now on we shall employ $\mathfrak{S}$-topologies on E'. Prominent choices for $\mathfrak{S}$ are the families $\mathcal{F}(E)$ and $\mathcal{K}(E)$ of finite and compact subsets E respectively. For every $S \in \mathfrak{S}$ let

$$p_S(a) := \sup\{|\langle x, a\rangle| : x \in S\}$$

whenever $a \in E'$. We know from Appendices B.7, B.8 that p_S is a seminorm on E'. The topologies generated in E' by the sets $\{p_S : S \in \mathfrak{S}\}$ for $\mathfrak{S}$ equal to $\mathcal{F}(E)$ and $\mathcal{K}(E)$ of simple and compact convergence will be denoted by $\sigma(E', E)$ and $\tau(E', E)$ respectively. Clearly

$$\sigma(E', E) \succ \tau(E', E)\,.$$

Properties 2.1.6 *Let $\mu, \nu \in M^{\mathrm{b}}(E)$ and $a, b \in E'$. Then*

2.1.6.1 $|\hat{\mu}(a)| \leqslant \hat{\mu}(0) = \mu(E)$.

2.1.6.2 $\hat{\mu}(-a) = \overline{\hat{\mu}(a)}$.

2.1.6.3 $|\hat{\mu}(a) - \hat{\mu}(b)|^2 \leqslant 2\hat{\mu}(0)\{\hat{\mu}(0)\,\mathrm{Re}\,\hat{\mu}(a - b)\}$.

2.1.6.4 $\hat{\mu}$ *is* $\tau(E', E)$-*continuous.*

2.1.6.5 *If H is a uniformly tight subset of $M^{\mathrm{b}}(E)$ then $\{\hat{\mu} : \mu \in H\}$ is $\tau(E', E)$-equicontinuous.*

2.1.6.6 *Suppose there exists $\delta > 0$ such that $\hat{\mu}(a) = 1$ for all $a \in E'$ with $\|a\| < \delta$. Then $\mu = \varepsilon_0$.*

2.1.6.7 $(\mu * \nu)^{\wedge} = \hat{\mu}\hat{\nu}$.

Proof. To show 2.1.6.1 we just consider the equalities

$$|\hat{\mu}(a)| = \left| \int e^{i\langle x, a \rangle} \, \mu(\mathrm{d}x) \right| \leqslant \int |e^{i\langle x, a \rangle}| \, \mu(\mathrm{d}x) = \int \mathbb{1}_E \, \mathrm{d}\mu = \mu(E) \,.$$

Property 2.1.6.2 follows from

$$\hat{\mu}(-a) = \int e^{-i\langle x, a \rangle} \, \mu(\mathrm{d}x) = \int \overline{e^{i\langle x, a \rangle}} \, \mu(\mathrm{d}x) = \overline{\int e^{i\langle x, a \rangle} \, \mu(\mathrm{d}x)} = \overline{\hat{\mu}(a)} \,.$$

As for the proof of 2.1.6.3 we first note that for $\alpha, \beta \in \mathbb{R}$ we have

$$\begin{aligned}
|e^{i\alpha} - e^{i\beta}|^2 &= |e^{i\beta}|^2 |e^{i(\alpha - \beta)} - 1|^2 \\
&= (e^{i(\alpha - \beta)} - 1)(e^{-i(\alpha - \beta)} - 1) = 2 - 2\cos(\alpha - \beta) \,.
\end{aligned}$$

Applying the Cauchy-Schwarz inequality we then obtain

$$\begin{aligned}
|\hat{\mu}(a) - \hat{\mu}(b)|^2 &= \left| \int (e^{i\langle x, a \rangle} - e^{i\langle x, b \rangle}) \, \mu(\mathrm{d}x) \right|^2 \\
&\leqslant \int |e^{i\langle x, a \rangle} - e^{i\langle x, b \rangle}|^2 \, \mu(\mathrm{d}x) \int \mathbb{1}_E^2 \, \mathrm{d}\mu \\
&= 2 \int (1 - \cos\langle x, a - b \rangle) \, \mu(\mathrm{d}x) \hat{\mu}(0) \\
&= 2\hat{\mu}(0) \left(\hat{\mu}(0) - \int \mathrm{Re}\, e^{i\langle x, a-b \rangle} \, \mu(\mathrm{d}x) \right) \\
&= 2\hat{\mu}(0)(\hat{\mu}(0) - \mathrm{Re}\, \hat{\mu}(a - b)) \,.
\end{aligned}$$

In order to show 2.1.6.4 given $\varepsilon > 0$ we choose $K \in \mathcal{K}(E)$ such that $\mu(\complement K) < \varepsilon/4$. Moreover, there exists $\delta > 0$ such that

$$|e^{is} - e^{it}| \leqslant \frac{\varepsilon}{2\mu(E)}$$

for all $s, t \in \mathbb{R}$ with $|s - t| < \delta$. Now suppose $a, b \in E'$ satisfy $p^K(a - b) < \delta$. Then

$$\begin{aligned}
|\hat{\mu}(a) - \hat{\mu}(b)| &= \left| \int (e^{i\langle x, a \rangle} - e^{i\langle x, b \rangle}) \, \mu(\mathrm{d}x) \right| \\
&\leqslant 2\mu(\complement K) + \int_K |e^{i\langle x, a \rangle} - e^{i\langle x, b \rangle}| \, \mu(\mathrm{d}x) < \varepsilon \,.
\end{aligned}$$

The proof of 2.1.6.5 is analogous to that of 2.1.6.4. As for 2.1.6.6 we apply 2.1.6.3 in order to deduce that

$$|\hat{\mu}(2a) - \hat{\mu}(a)|^2 \leqslant 2(1 - \operatorname{Re}\hat{\mu}(a))$$

for all $a \in E'$. Now for $a \in E'$ and $\delta > 0$ there exists $k \geqslant 1$ satisfying $\|a/2^k\| \leqslant \delta$. By assumption $\hat{\mu}(a/2^k) = 1$ and consequently

$$1 = \hat{\mu}\left(2\frac{a}{2^k}\right) = \cdots = \hat{\mu}\left(2^k\frac{a}{2^k}\right) = \hat{\mu}(a)$$

and hence $\mu = \varepsilon_0$.

Finally, 2.1.6.7 is immediate from the following chain of equalities:

$$\begin{aligned}
(\mu * \nu)^{\wedge}(a) &= \int e^{i\langle x,a\rangle}(\mu * \nu)(dx) \\
&= \iint e^{i\langle x+y,a\rangle}\,\mu(dx)\,\nu(dy) \\
&= \int e^{i\langle y,a\rangle}\left(\int e^{i\langle x,a\rangle}\,\mu(dx)\right)\nu(dy) \\
&= \hat{\mu}(a)\hat{\nu}(a)\,.
\end{aligned}$$

$\square$

Given $\delta > 0$ we now consider the set

$$V_\delta := \{a \in E' : \|a\| \leqslant \delta\}$$

which by Appendix B.9.3 is equicontinuous, hence by Appendix B.14.1 $\sigma(E', E)$-compact. Moreover,

$$\operatorname{Res}_{V_\delta}\sigma(E', E) = \operatorname{Res}_{V_\delta}\tau(E', E)\,.$$

From Appendix B.14.2 we infer that, if E is assumed to the separable, V_δ is metrizable with respect to the topologies $\sigma(E', E)$ and $\tau(E', E)$.

Let $C(V_\delta) := C(V_\delta, \tau(E', E), \mathbb{C})$ denote the Banach space of all $\tau(E', E)$-continuous complex-valued functions on V_δ, together with the sup norm $\|\cdot\|_\infty$. Then for any $\mu \in M^{\mathrm{b}}(E)$ we obtain

Properties 2.1.7 of the Fourier transform.

2.1.7.1 $\operatorname{Res}_{V_\delta}\hat{\mu} \in C(V_\delta)$.

2.1.7.2 $\|\operatorname{Res}_{V_\delta}\hat{\mu}\| \leqslant \mu(E)$.

In the following we shall identify $\operatorname{Res}_{V_\delta}\hat{\mu}$ with $\hat{\mu}$.

Theorem 2.1.8 *For every τ_{w}-relatively compact subset H of $M^{\mathrm{b}}(E)$ the set $\hat{H}$ is relatively compact in $C(V_\delta)$.*

Proof. From Prohorov's theorem 1.3.7 we infer that H is uniformly tight. Then Property 2.1.7.2 implies that $\hat{H}$ is bounded in $C(V_\delta)$, and by Property 2.1.6.5, $\hat{H}$ is equicontinuous. But then the Arzelà–Ascoli theorem yields the assertion. $\square$

Theorem 2.1.9 (Continuity of the Fourier transform)
Let $(\mu_n)_{n\geqslant 1}$ be a sequence of measures in $M^{\mathrm{b}}(E)$ and let $\mu \in M^{\mathrm{b}}(E)$. The following statements are equivalent:

  (i) *$(\mu_n)_{n\geqslant 1}$ converges with respect to τ_{w}.*
 (ii) *$(\mu_n)_{n\geqslant 1}$ is τ_{w}-relatively compact, and for every $\delta > 0$ the sequence $(\hat{\mu}_n)_{n\geqslant 1}$ converges uniformly on V_δ.*
(iii) *$(\mu_n)_{n\geqslant 1}$ is τ_{w}-relatively compact, and for every $a \in E'$ the sequence $(\hat{\mu}_n(a))_{n\geqslant 1}$ converges in $\mathbb{C}$.*

If in (i) we assume in addition that $\tau_{\mathrm{w}}-\lim_{n\to\infty} \mu_n = \mu$ then in (iii) we have

$$\lim_{n\to\infty} \hat{\mu}_n(a) = \hat{\mu}(a)$$

for all $a \in E'$.

Proof. (i) $\Longrightarrow$ (ii). From the τ_{w}-convergence of the sequence $(\mu_n)_{n\geqslant 1}$ follows its τ_{w}-relative compactness. Then Theorem 2.1.8 implies that $(\hat{\mu}_n)_{n\geqslant 1}$ is relatively compact in $C(V_\delta)$ for every $\delta > 0$. But from $\tau_{\mathrm{w}}-\lim_{n\to\infty} \mu_n = \mu$ it follows that

$$\lim_{n\to\infty} \hat{\mu}_n(a) = \hat{\mu}(a)$$

for all $a \in E'$, since $\operatorname{Re} \mathrm{e}^{\mathrm{i}\langle \cdot, a\rangle}$ and $\operatorname{Im} \mathrm{e}^{\mathrm{i}\langle \cdot, a\rangle}$ belong to $C^{\mathrm{b}}(E)$ whenever $a \in E'$. Note that this argument takes care of the last statement of the theorem. We conclude that all the accumulation points of $(\hat{\mu}_n)_{n\geqslant 1}$ in $C(V_\delta)$ coincide with $\operatorname{Res}_{V_\delta} \hat{\mu}$, and hence that $\lim_{n\to\infty} \mu_n = \mu$ uniformly on V_δ.

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i). Let

$$\varphi(a) := \lim_{n\to\infty} \hat{\mu}_n(a)$$

for all $a \in E'$. Moreover, let μ_0 be an accumulation point of $(\mu_n)_{n\geqslant 1}$, which means that $\tau_{\mathrm{w}}-\lim_{k\to\infty} \mu_{n_k} = \mu_0$ for some subsequence $(\mu_{n_k})_{k\geqslant 1}$ of $(\mu_n)_{n\geqslant 1}$. But then

$$\hat{\mu}_0(a) = \lim_{k\to\infty} \hat{\mu}_{n_k}(a) = \varphi(a)$$

for all $a \in E'$. The uniqueness theorem 2.1.4 implies that $(\mu_n)_{n \geqslant 1}$ admits only one accumulation point, and consequently $(\mu_n)_{n \geqslant 1}$ τ_{w}-converges to it. □

We are now in a position to study the *logarithm of the Fourier transform*.

Theorem 2.1.10 *Consider* $\mu \in M^1(E)$ *satisfying* $\hat{\mu}(a) \neq 0$ *for all* $a \in E'$. *Then there exists exactly one complex-valued function* h *on* E' *admitting the following properties:*

(i) $h(0) = 0$.
(ii) h *is norm-continuous.*
(iii) *For each* $a \in E'$,

$$\hat{\mu}(a) = \exp h(a) .$$

In the sequel we shall write $\operatorname{Log} \hat{\mu}$ instead of h.

Example 2.1.11 For every $x \in E$

$$\operatorname{Log} \hat{\varepsilon}_x = \mathrm{i}\langle x, \cdot \rangle .$$

Proof of Theorem 2.1.10. Let $\delta > 0$. Since V_δ is compact and $\hat{\mu}$ is $\tau(E', E)$-continuous by Property 2.1.6.4, we have that

$$\alpha := \inf\{|\hat{\mu}(a)| : a \in V_\delta\} > 0 .$$

It follows from Properties 2.1.6.3 and 2.1.6.4 that $\hat{\mu}$ is uniformly continuous on E' (with respect to the norm in E'), and hence there exists $\varepsilon > 0$ such that

$$|\hat{\mu}(a) - \hat{\mu}(b)| < \frac{\alpha}{2}$$

for all $a, b \in E'$ with $\|a - b\| < \varepsilon$. Let $\{t_0, t_1, \ldots, t_n\}$ be a partition of $[0, 1]$ of mesh size less than ε/δ. Then for all $a \in V_\delta$ and $j \in \{1, 2, \ldots, n\}$ we have

$$\|t_j a - t_{j-1} a\| = (t_j - t_{j-1})\|a\| < \frac{\varepsilon}{\delta}\delta = \varepsilon$$

and therefore

$$|\hat{\mu}(t_j a) - \hat{\mu}(t_{j-1} a)| < \frac{\alpha}{2} .$$

Consequently

$$\left| \frac{\hat{\mu}(t_j a)}{\hat{\mu}(t_{j-1} a)} - 1 \right| < \frac{\alpha}{2|\hat{\mu}(t_{j-1} a)|} \leqslant \frac{1}{2}$$

whenever $a \in V_\delta$ and $j \in \{1, 2, \ldots, n\}$. Now, let Log denote the main branch of the complex logarithm. We define

$$h_\delta(a) := \sum_{j=1}^n \mathrm{Log} \, \frac{\hat{\mu}(t_j a)}{\hat{\mu}(t_{j-1} a)}$$

for all $a \in V_\delta$. Then h_δ is a complex-valued function on V_δ with the following properties:

(i') $h_\delta(0) = 0$.
(ii') h_δ is $\tau(E', E)$-continuous, and hence also $\|\cdot\|_\infty$-continuous.
(iii') For all $a \in V_\delta$

$$\hat{\mu}(a) = \exp(h_\delta(a)).$$

We shall show that h_δ is uniquely determined by properties (i') to (iii'). In fact, let g_δ be another complex-valued function satisfying these three properties. Then for each $a \in V_\delta$ there exists $k(a) \in \mathbb{Z}$ satisfying

$$h_\delta(a) - g_\delta(a) = 2\pi \mathrm{i} k(a)$$

(by (iii')). But k is $\|\cdot\|_\infty$-continuous on V_δ (by (ii')) and satisfies $k(0) = 0$ (by (i')). Since V_δ is $\|\cdot\|_\infty$-connected, $k(V_\delta)$ is a connected subset of $\mathbb{Z}$. This implies $k = 0$, and hence $h_\delta = g_\delta$.

Now, choose $0 < \delta_1 < \delta_2$. By the above discussion we have $\mathrm{Res}_{V_{\delta_1}} h_{\delta_2} = h_{\delta_1}$, and hence $h(a) := h_\delta(a)$ for all $a \in V_\delta$ defines the desired function on E' which is certainly uniquely determined by properties (i) to (iii).   □

Corollary 2.1.12 *For every $\delta > 0$ $\mathrm{Log} \, \hat{\mu}$ is $\tau(E', E)$-continuous on V_δ. In particular*

$$\sup\{|\mathrm{Log} \, \hat{\mu}(a)| : \|a\| \leqslant \delta\} < \infty.$$

Proof. The first assertion follows from the representation

$$(\mathrm{Res}_{V_\delta} \hat{\mu})(a) = \sum_{j=1}^n \mathrm{Log} \, \frac{\hat{\mu}(t_j a)}{\hat{\mu}(t_{j-1} a)}$$

valid for all $a \in V_\delta$, together with the $\tau(E', E)$-continuity of $\hat{\mu}$ (by Property 2.1.6.4). The second assertion is a consequence of the $\tau(E', E)$-compactness of V_δ.   □

Theorem 2.1.13 Let $H \subset M^1(E)$ such that $\hat{\mu}(a) \neq 0$ for all $a \in E'$, $\mu \in H$. Suppose there exist $\alpha, \delta > 0$ satisfying $|\hat{\mu}(a)| \geqslant \alpha$ for all $a \in V_\delta$, $\mu \in H$, and such that $\hat{H}$ is equicontinuous on V_δ with respect to $\tau(E', E)$. Then

$$\{\operatorname{Res}_{V_\delta} \operatorname{Log} \hat{\mu} : \mu \in H\}$$

is relatively compact in $C(V_\delta)$.

Proof. From Property 2.1.6.3 together with the assumption, $\hat{H}$ is uniformly equicontinuous on V_δ with respect to the norm $\|\cdot\|_\infty$. Therefore there exists a partition $\{t_0, t_1, \ldots, t_n\}$ of $[0, 1]$ with sufficiently small mesh such that

$$\left| \frac{\hat{\mu}(t_j a)}{\hat{\mu}(t_{j-1} a)} - 1 \right| \leqslant \frac{1}{2}$$

for all $j \in \{1, 2, \ldots, n\}$ and

$$\operatorname{Log} \hat{\mu}(a) = \sum_{j=1}^{n} \operatorname{Log} \frac{\hat{\mu}(t_j a)}{\hat{\mu}(t_{j-1} a)}$$

for all $a \in V_\delta$, $\mu \in H$ (compare the proof of Theorem 2.1.10). But then

$$\{\operatorname{Res}_{V_\delta} \operatorname{Log} \hat{\mu} : \mu \in H\}$$

is bounded and equicontinuous with respect to $\tau(E', E)$, and the Arzelà-Ascoli theorem yields the assertion. □

Lemma 2.1.14 Let $\delta > 0$ be given. For every $x \in E$ define $J(x)(a) := \langle x, a \rangle$ whenever $a \in V_\delta$. Then $J(x) \in C(V_\delta)$ for all $x \in E$, and $\frac{1}{\delta} J$ is a linear isometry from E onto a closed subspace of $C(V_\delta)$.

Proof. We first note that $J(x)$ is $\tau(E', E)$-continuous. Moreover, J is a linear mapping from E into $C(V_\delta)$. By Appendix B.14.3

$$\|J(x)\|_\delta := \sup\{|\langle x, a \rangle| : a \in V_\delta\}$$
$$= \delta \sup\{|\langle x, a \rangle| : \|a\| \leqslant 1\} = \delta\|x\|$$

for all $x \in E$. Therefore $\frac{1}{\delta} J$ is a linear isometry from E into $C(V_\delta)$. In particular, $\frac{1}{\delta} J(E) = J(E)$ is closed in $C(V_\delta)$. □

Corollary 2.1.15 to Theorem 2.1.13.
Let $M \subset E$ be such that

$$\{\operatorname{Res}_{V_\delta} \hat{\varepsilon}_x : x \in M\}$$

is relatively compact in $C(V_\delta)$ (for some $\delta > 0$). Then M is relatively compact in E. Furthermore $x \mapsto \operatorname{Res}_{V_\delta} \hat{\varepsilon}_x$ is a homeomorphism from E onto a closed subset of $C(V_\delta)$.

Proof. Firstly $|\hat{\varepsilon}_x(a)| = 1$ for all $x \in M$, $a \in E'$. By assumption $\{\hat{\varepsilon}_x : x \in M\}$ is equicontinuous on V_δ with respect to $\tau(E', E)$. Now we infer from the theorem that

$$\{\mathrm{Res}_{V_\delta} \mathrm{Log}\, \hat{\varepsilon}_x : x \in M\}$$

is relatively compact in $C(V_\delta)$, and hence $\{J(x) \in M\}$ is relatively compact in $C(V_\delta)$. Lemma 2.1.14 yields the assertion. $\qquad\square$

Corollary 2.1.16 *Let $(\mu_k)_{k \geqslant 1}$ be a sequence of measures with τ_w-limit $\mu \in M^1(E)$ such that $\hat{\mu}(a) \neq 0$, $\hat{\mu}_k(a) \neq 0$ for all $a \in E$, $k \geqslant 1$. Then*

$$\lim_{k \to \infty} \mathrm{Log}\, \hat{\mu}_k = \mathrm{Log}\, \hat{\mu}$$

uniformly on bounded subsets of E'.

Proof. Let $\delta > 0$ be given. As in the proof of Theorem 2.1.10 we see that

$$\alpha := \frac{1}{2} \inf\{\hat{\mu}(a) : a \in V_\delta\} > 0.$$

Applying the continuity theorem 2.1.9 we obtain the existence of $k_0 \in \mathbb{N}$ such that $|\hat{\mu}_k(a)| \geqslant \alpha$ for all $a \in V_\delta$, $k \geqslant k_0$. We now infer from Prohorov's theorem 1.3.7 that $\{\mu_k : k \geqslant k_0\}$ is uniformly tight, and from Property 2.1.6.5 that $\{\hat{\mu}_k : k \geqslant k_0\}$ is equicontinuous with respect to $\tau(E', E)$. But Theorem 2.1.13 implies that

$$\{\mathrm{Res}_{V_\delta} \mathrm{Log}\, \hat{\mu}_k : k \geqslant k_0\}$$

is relatively compact in $C(V_\delta)$. On the other hand the representations of $\mathrm{Log}\, \hat{\mu}_k$ and $\mathrm{Log}\, \hat{\mu}$ on V_δ (see the proof of Theorem 2.1.13) together with the continuity theorem 2.1.9 yield the limit relationship

$$\lim_{k \to \infty} \mathrm{Log}\, \hat{\mu}_k(a) = \mathrm{Log}\, \hat{\mu}(a)$$

for all $a \in V_\delta$. Since $\{\mathrm{Res}_{V_\delta} \mathrm{Log}\, \hat{\mu}_k : k \geqslant k_0\}$ is relatively compact, this convergence is uniform on V_δ. $\qquad\square$

2.2   Shift compact sets of probability measures

We start with two useful results, following from properties of the Fourier transform $\mu \mapsto \hat{\mu}$ on the topological semigroup $(M^1(E), *, \tau_{\mathrm{w}})$. Here E is a given separable Banach space so that, by Theorem 1.4.9, $(M^1(E), *, \tau_{\mathrm{w}})$ is a metric semigroup.

Lemma 2.2.1 *Suppose $\mu \in M^1(E)$ has the property that $(\mu^n)_{n \geqslant 1}$ is τ_{w}-relatively compact. Then $\mu = \varepsilon_0$.*

Proof.   From Prohorov's theorem 1.3.7 together with Property 2.1.6.5 (of the Fourier transform) we conclude that the set $\{(\mu^n)^{\wedge} : n \in \mathbb{N}\}$ is equicontinuous with respect to $\tau(E', E)$. But we have $(\mu^n)^{\wedge} = (\hat{\mu})^n$ for all $n \in \mathbb{N}$. Therefore there exists $\delta > 0$ such that

$$\left| 1 - (\hat{\mu}(a))^n \right| \leqslant \frac{1}{2}$$

whenever $a \in V_\delta$, $n \in \mathbb{N}$. Consequently $\hat{\mu}(a) = 1$ for all $a \in V_\delta$ as we shall see now. Indeed, let $z \in \mathbb{C}$ with $|1 - z^n| \leqslant 1/2$ for all $n \in \mathbb{N}$. Then $\operatorname{Re} z^n \geqslant 1/2$ for all $n \in \mathbb{N}$, and

$$\frac{1}{2} \geqslant |1 - z^n| = |1 - z||1 + z + \cdots + z^{n-1}|$$
$$\geqslant |1 - z|\left(1 + \operatorname{Re} z + \cdots + \operatorname{Re} z^{n-1}\right) \geqslant |1 - z|\frac{n}{2},$$

so that $|1 - z| \leqslant 1/n$ for all $n \in \mathbb{N}$, and consequently $z = 1$.

The assertion now follows from $\hat{\mu}(a) = 1$ for all $a \in V_\delta$ with the help of Property 2.1.6.6 of the Fourier transform. $\square$

Lemma 2.2.2 *Let $\mu, \nu \in M^1(E)$ such that $\mu * \nu = \nu$. Then $\nu = \varepsilon_0$.*

Proof.   Properties 2.1.6.1 and 2.1.6.4 of the Fourier transform give the existence of $\delta > 0$ satisfying $\hat{\mu}(a) \neq 0$ for all $a \in V_\delta$. Moreover, $\hat{\mu}(a)\hat{\nu}(a) = \hat{\mu}(a)$ and hence $\hat{\nu}(a) = 1$ for all $a \in V_\delta$. We now apply Property 2.1.6.6 of the Fourier transform in order to obtain $\nu = \varepsilon_0$. $\square$

Theorem 2.2.3 *Let $(\mu_n)_{n \geqslant 1}$, $(\nu_n)_{n \geqslant 1}$ be sequences in $M^1(E)$ such that $\{\mu_n * \nu_n : n \geqslant 1\}$ and $\{\mu_n : n \geqslant 1\}$ are both τ_{w}-relatively compact. Then $\{\nu_n : n \geqslant 1\}$ is also τ_{w}-relatively compact.*

Proof.   Let $\varepsilon > 0$. By Prohorov's theorem 1.3.7 there exists $K \in \mathcal{K}(E)$ such that for all $n \geqslant 1$

$$\mu_n * \nu_n(K) \geqslant 1 - \varepsilon \qquad \text{and} \qquad \mu_n(K) \geqslant 1 - \varepsilon.$$

But

$$1 - \varepsilon \leqslant \mu_n * \nu_n(K) = \int \nu_n(K - x)\,\mu_n(\mathrm{d}x)$$

$$\leqslant \int_K \nu_n(K - x)\,\mu_n(\mathrm{d}x) + \varepsilon \leqslant \nu_n(K - K) + \varepsilon\,,$$

so that $\nu_n(K - K) \geqslant 1 - 2\varepsilon$ whenever $n \geqslant 1$. One more application of Prohorov's theorem 1.3.7 gives that $\{\nu_n : n \geqslant 1\}$ is τ_{w}-relatively compact. $\square$

Corollary 2.2.4 *Suppose that*

$$\tau_{\mathrm{w}} - \lim_{n \to \infty} \mu_n = \mu \in M^1(E)$$

and

$$\tau_{\mathrm{w}} - \lim_{n \to \infty} \mu_n * \nu_n = \lambda \in M^1(E)\,.$$

Moreover, assume $\hat{\mu}(a) \neq 0$ for all $a \in E'$. Then there exists $\nu \in M^1(E)$ such that

$$\tau_{\mathrm{w}} - \lim_{n \to \infty} \nu_n = \nu$$

and $\mu * \nu = \lambda$.

Proof.   Theorem 2.2.3 implies that $\{\nu_n : n \geqslant 1\}$ is τ_{w}-relatively compact. Furthermore the continuity theorem 2.1.9 implies that for all $a \in E'$

$$\lim_{n \to \infty} \hat{\nu}_n(a) = \lim_{n \to \infty} \frac{(\mu_n * \nu_n)^{\wedge}(a)}{\hat{\mu}_n(a)} = \frac{\hat{\lambda}(a)}{\hat{\mu}(a)}\,.$$

Another application of the continuity theorem 2.1.9 yields

$$\tau_{\mathrm{w}} - \lim_{n \to \infty} \nu_n = \nu \in M^1(E)\,.$$

But then

$$\hat{\nu}(a)\hat{\mu}(a) = \hat{\lambda}(a)$$

or

$$(\mu * \nu)^{\wedge}(a) = \hat{\lambda}(a)$$

for all $a \in E'$ which, by the uniqueness theorem 2.1.4, gives the desired result. $\square$

Remark 2.2.5 The assumption made in the corollary that $\hat{\mu}(a) \neq 0$ for all $a \in E'$ cannot be dropped (without replacement). In fact, from Lukacs [56], Theorem 5.1.1, we know that there exist $\mu, \nu_1, \nu_2 \in M^1(\mathbb{R})$ with $\nu_1 \neq \nu_2$ such that $\mu * \nu_1 = \mu * \nu_2$. The sequence $\{\mu * \nu_1, \mu * \nu_2, \mu * \nu_1, \dots\}$ obviously converges with respect to τ_{w}, but the sequence $\{\nu_1, \nu_2, \nu_1, \dots\}$ does not.

Definition 2.2.6 We refer to $H \subset M^1(E)$ as being **relatively shift compact** if for every $\mu \in H$ there exists $x_\mu \in E$ such that $\{\mu * \varepsilon_{x_\mu} : \mu \in H\}$ is τ_{w}-relatively compact.

We note by Prohorov's theorem 1.3.7 that relative shift compactness of a subset H is equivalent to the property that for every $\varepsilon > 0$ there exists $K \in \mathcal{K}(E)$ such that

$$\mu(K - x_\mu) = \mu * \varepsilon_{x_\mu}(K) \geqslant 1 - \varepsilon$$

for all $\mu \in H$, which motivates the use of the alternative terminology *shift tightness*.

Let $(\mu_n)_{n \geqslant 1}$ be a relatively shift compact sequence in $M^1(E)$, so that there exists a sequence $(x_n)_{n \geqslant 1}$ in E such that

$$\{\mu_n * \varepsilon_{x_n} : n \in \mathbb{N}\}$$

is τ_{w}-relatively compact. We refer to $(x_n)_{n \geqslant 1}$ as a **centralizing sequence** for $(\mu_n)_{n \geqslant 1}$ (equivalently, $(x_n)_{n \geqslant 1}$ **centralizes** $(\mu_n)_{n \geqslant 1}$).

Theorem 2.2.7 *Let $(\mu_n)_{n \geqslant 1}$, $(\nu_n)_{n \geqslant 1}$ be sequences in $M^1(E)$ such that $(\mu_n * \nu_n)_{n \geqslant 1}$ is τ_{w}-relatively compact. Then both $(\mu_n)_{n \geqslant 1}$ and $(\nu_n)_{n \geqslant 1}$ are relatively shift compact. Moreover, if $(x_n)_{n \geqslant 1}$ is a centralizing sequence for $(\mu_n)_{n \geqslant 1}$, then $(-x_n)_{n \geqslant 1}$ is a centralizing sequence for $(\nu_n)_{n \geqslant 1}$.*

Proof. Choose $(\delta_k)_{k \geqslant 1}$, $(\varepsilon_k)_{k \geqslant 1} \subset]0, \infty[$ with $\delta_1 < 1$, $\delta_k \downarrow 0$ and

$$\sum_{k \geqslant 1} \frac{\varepsilon_k}{\delta_k} \leqslant \frac{1}{2}.$$

By assumption for every $k \in \mathbb{N}$ there exists $K_k \in \mathcal{K}(E)$ satisfying

$$\mu_n * \nu_n(K_k) \geqslant 1 - \varepsilon_k$$

for all $n \in \mathbb{N}$. Let

$$B_{n,k} := \{x \in E : \mu_n(K_k - x) > 1 - \delta_k\}$$

and put

$$B_n := \bigcap_{k \geqslant 1} B_{n,k} \, .$$

We have for all $k \geqslant 1$

$$\varepsilon_k \geqslant \mu_n * \nu_n(\complement K_k) = \int \mu_n(\complement K_k - x) \, \nu_n(\mathrm{d}x)$$

$$= \int \mu_n(\complement(K_k - x)) \, \nu_n(\mathrm{d}x)$$

$$\geqslant \int_{\complement B_{n,k}} (1 - \mu_n(K_k - x)) \, \nu_n(\mathrm{d}x)$$

$$\geqslant \nu_n(\complement B_{n,k})(1 - (1 - \delta_k)) = \delta_k \nu_n(\complement B_{n,k}) \, ,$$

so that

$$\nu_n(\complement B_n) \leqslant \sum_{k \geqslant 1} \nu_n(\complement B_{n,k}) \leqslant \sum_{k \geqslant 1} \frac{\varepsilon_k}{\delta_k} \leqslant \frac{1}{2}$$

and thus for all $n \geqslant 1$

$$\nu_n(B_n) \geqslant \frac{1}{2} \, .$$

In particular $B_n \neq \emptyset$ and so there exists $x_n \in E$ with

$$\mu_n(K_k - x_n) > 1 - \delta_k$$

for all $k \in \mathbb{N}$. This implies the τ_w-relative compactness of $(\mu_n * \varepsilon_{x_n})_{n \geqslant 1}$.

But since

$$\mu_n * \nu_n = (\mu_n * \varepsilon_{x_n}) * (\nu_n * \varepsilon_{-x_n})$$

for all $n \geqslant 1$, Theorem 2.2.3 implies the τ_w-relative compactness of $(\nu * \varepsilon_{-x_n})_{n \geqslant 1}$. $\qquad\qquad\Box$

Corollary 2.2.8 *The following statements are equivalent:*

(i) $(\mu_n * \nu_n)_{n \geqslant 1}$ *is shift tight.*
(ii) $(\mu_n)_{n \geqslant 1}$ *and* $(\nu_n)_{n \geqslant 1}$ *are shift tight.*

Proof. (i) $\Rightarrow$ (ii). Let $(x_n)_{n \geqslant 1}$ be a centralizing sequence for $(\mu_n * \nu_n)_{n \geqslant 1}$. Then Theorem 2.2.7 implies that $(\mu_n)_{n \geqslant 1}$ and $(\nu_n * \varepsilon_{x_n})_{n \geqslant 1}$ are shift tight, and consequently $(\nu_n)_{n \geqslant 1}$ is also shift tight.

(ii) $\Rightarrow$ (i). Let $(x_n)_{n \geqslant 1}, (y_n)_{n \geqslant 1}$ be centralizing sequences for $(\mu_n)_{n \geqslant 1}$, $(\nu_n)_{n \geqslant 1}$ respectively. Since

$$(\mu * \varepsilon_{x_n}) * (\nu * \varepsilon_{y_n}) = (\mu_n * \nu_n) * \varepsilon_{x_n + y_n}$$

for all $n \geqslant 1$ and since the convolution $(\mu, \nu) \mapsto \mu * \nu$ is continuous on $M^1(E) \times M^1(E)$ the sequence $(x_n + y_n)_{n \geqslant 1}$ centralizes $(\mu_n * \nu_n)_{n \geqslant 1}$. $\Box$

Theorem 2.2.9 *For every shift tight sequence* $(\mu_n)_{n \geqslant 1}$ *in* $M^1(E)$ *the following statements are equivalent:*

(i) $(\mu_n)_{n \geqslant 1}$ *is* τ_{w}-*relatively compact.*
(ii) *Each centralizing sequence for* $(\mu_n)_{n \geqslant 1}$ *is relatively compact.*
(iii) *There is a relatively compact centralizing sequence for* $(\mu_n)_{n \geqslant 1}$.

Proof.   (i) $\Rightarrow$ (ii). Let $(x_n)_{n \geqslant 1}$ be a centralizing sequence for $(\mu_n)_{n \geqslant 1}$. Then the sequences $(\mu * \varepsilon_{x_n})_{n \geqslant 1}$ and $(\mu_n)_{n \geqslant 1}$ are both τ_{w}-relatively compact. By Theorem 2.2.3 it follows that $(\varepsilon_{x_n})_{n \geqslant 1}$ is τ_{w}-relatively compact, and hence $(x_n)_{n \geqslant 1}$ is relatively compact (see Theorem 1.2.11).

(ii) $\Rightarrow$ (iii)   is clear.

(iii) $\Rightarrow$ (i).   Let $(x_n)_{n \geqslant 1}$ denote a relatively compact centralizing sequence for $(\mu_n)_{n \geqslant 1}$. Then the sequences $(\mu * \varepsilon_{x_n})_{n \geqslant 1}$ and $(\varepsilon_{-x_n})_{n \geqslant 1}$ are τ_{w}-relatively compact. But

$$\mu_n = (\mu_n * \varepsilon_{x_n}) * \varepsilon_{-x_n}$$

for every $n \geqslant 1$;   this together with the continuity of the convolution in $M^1(E)$ yields the desired conclusion. $\square$

Corollary 2.2.10 *Given a centralizing sequence* $(x_n)_{n \geqslant 1}$ *for* $(\mu_n)_{n \geqslant 1}$ *and an arbitrary sequence* $(y_n)_{n \geqslant 1}$ *in* E *the following statements are equivalent:*

(i) $(y_n)_{n \geqslant 1}$ *centralizes* $(\mu_n)_{n \geqslant 1}$.
(ii) $(x_n - y_n)_{n \geqslant 1}$ *is relatively compact.*

Proof.   For every $n \geqslant 1$ we have

$$(\mu_n * \varepsilon_{y_n}) * \varepsilon_{x_n - y_n} = \mu_n * \varepsilon_{x_n} .$$

Therefore $(\mu_n * \varepsilon_{y_n})_{n \geqslant 1}$ is shift tight and $(x_n - y_n)_{n \geqslant 1}$ centralizes $(\mu_n * \varepsilon_{y_n})_{n \geqslant 1}$. The equivalence now follows from Theorem 2.2.9. $\square$

Corollary 2.2.11 *Let* $(\mu_n)_{n \geqslant 1}$, $(x_n)_{n \geqslant 1}$ *be sequences in* $M^1(E)$ *and* E *respectively satisfying*

$$\tau_{\mathrm{w}} - \lim_{n \to \infty} \mu_n = \mu \in M^1(E)$$

and

$$\tau_{\mathrm{w}} - \lim_{n \to \infty} \mu_n * \varepsilon_{x_n} = \nu \in M^1(E) .$$

Then there exists $x := \lim_{n \to \infty} x_n$, *and* $\nu = \mu * \varepsilon_x$.

Proof. From Theorem 2.2.9 we conclude that $(x_n)_{n \geqslant 1}$ is relatively compact. Let x, y be accumulation points of $(x_n)_{n \geqslant 1}$. Since the convolution in $M^1(E)$ is continuous,

$$\mu * \varepsilon_x = \nu = \mu * \varepsilon_y \, .$$

Putting $z := x - y$ we obtain $\mu * \varepsilon_z = \mu$. But now Lemma 2.2.2 applies to give $z = 0$, so that $x = y$. Consequently $\lim_{n \to \infty} x_n = x$, and this takes care of the assertion. $\qquad \square$

Theorem 2.2.12 *Let $(\mu_n)_{n \geqslant 1}$ be a shift tight sequence in $M^1(E)$. The following statements are equivalent:*

(i) *$(\mu_n)_{n \geqslant 1}$ is τ_w-relatively compact.*
(ii) *For some (each) $\delta > 0$ the sequence $(\mathrm{Res}_{V_\delta} \hat{\mu}_n)_{n \geqslant 1}$ is relatively compact in $C(V_\delta)$.*
(iii) *For some (each) $\delta > 0$ the sequence $(\mathrm{Res}_{V_\delta} \hat{\mu}_n)_{n \geqslant 1}$ is equicontinuous in 0 with respect to $\tau(E', E)$.*

Proof. (ii) $\Leftrightarrow$ (iii). Property 2.1.6.3 of the Fourier transform together with $|\hat{\mu}_n| \leqslant 1$ for all $n \geqslant 1$ and the Arzelà-Ascoli theorem imply the assertion.

(i) $\Rightarrow$ (ii) follows from Prohorov's theorem 1.3.7 in conjunction with Property 2.1.6.5 of the Fourier transform.

(iii) $\Rightarrow$ (i). From Theorem 2.2.9 we see that it suffices to show the existence of a relatively compact centralizing sequence for $(\mu_n)_{n \geqslant 1}$. Let therefore $(x_n)_{n \geqslant 1}$ be a centralizing sequence for $(\mu_n)_{n \geqslant 1}$. By assumption and Property 2.1.6.5 of the Fourier transform we have, for every $0 < \varepsilon < 2$, $K \in \mathcal{K}(E)$ and $\eta > 0$ such that

$$|1 - \hat{\mu}_n(a)| \leqslant \frac{\varepsilon}{2}$$

and

$$\left| 1 - \hat{\mu}_n(a) \mathrm{e}^{\mathrm{i}\langle x_n, a \rangle} \right| \leqslant \frac{\varepsilon}{2}$$

for all $a \in V_\delta$ with $p^K(a) < \eta$ and all $n \geqslant 1$. For such a and all $n \geqslant 1$ we then obtain the inequalities

$$\left| \hat{\mu}_n(a) - \hat{\mu}_n(a) \mathrm{e}^{\mathrm{i}\langle x_n, a \rangle} \right| \leqslant \varepsilon$$

and

$$|\hat{\mu}_n(a)| \geqslant 1 - \frac{\varepsilon}{2} \, ,$$

and hence that

$$|1 - \mathrm{e}^{\mathrm{i}\langle x_n, a \rangle}| \leqslant \frac{1}{1 - \frac{\varepsilon}{2}} \left| \hat{\mu}_n(a) - \hat{\mu}_n(a) \mathrm{e}^{\mathrm{i}\langle x_n, a \rangle} \right| \leqslant \frac{\varepsilon}{1 - \frac{\varepsilon}{2}} \, .$$

Consequently $(\mathrm{Res}_{V_\delta} \hat{\varepsilon}_{x_n})_{n \geqslant 1}$ is equicontinuous in 0, and hence relatively compact in $C(V_\delta)$ by Property 2.1.6.3 of the Fourier transform. The homeomorphism $x \mapsto \mathrm{Res}_{V_\delta} \hat{\varepsilon}_x$ between E and $C(V_\delta)$ provides the final step of the proof. $\qquad\square$

Corollary 2.2.13 *Let* $(\mu_n)_{n \geqslant 1}$, $(x_n)_{n \geqslant 1}$ *be sequences in* $M^1(E)$, E *respectively satisfying*

$$\tau_{\mathrm{w}} - \lim_{n \to \infty} \mu_n * \varepsilon_{x_n} =: \nu \in M^1(E) \,.$$

Then the following statements are equivalent:

(i) $(\mu_n)_{n \geqslant 1}$ *is* τ_{w}*-convergent.*
(ii) *For some (each)* $\delta > 0$, $(\hat{\mu}_n)_{n \geqslant 1}$ *converges uniformly on* V_δ.

Proof. (i) $\Rightarrow$ (ii). This follows directly from the continuity theorem 2.1.9.

(ii) $\Rightarrow$ (i). By assumption $(\mu_n)_{n \geqslant 1}$ is shift tight, and by Theorem 2.2.12 $(\mu_n)_{n \geqslant 1}$ is τ_{w}-relatively compact. We now employ Theorem 2.2.3 and the homeomorphism $x \mapsto \varepsilon_x$ in order to obtain that $(x_n)_{n \geqslant 1}$ is relatively compact in E. Let x, y be accumulation points of $(x_n)_{n \geqslant 1}$. Moreover, let

$$f := \lim_{n \to \infty} \mathrm{Res}_{V_\delta} \hat{\mu}_n \in C(V_\delta)$$

(for some $\delta > 0$). Then by the continuity theorem 2.1.9

$$f(a) \mathrm{e}^{\mathrm{i}\langle x, a \rangle} = \hat{\nu}(a) = f(a) \mathrm{e}^{\mathrm{i}\langle y, a \rangle}$$

for all $a \in V_\delta$. But now $\hat{\nu}(0) = 1$, and $\hat{\nu}$ is $\| \cdot \|$-continuous by Property 2.1.6.4 of the Fourier transform, hence there exists $\delta' \in {]0, \delta]}$ with $\hat{\nu}(a) \neq 0$ for all $a \in V_{\delta'}$. It follows that

$$\mathrm{e}^{\mathrm{i}\langle x - y, a \rangle} = 1$$

for all $a \in V_{\delta'}$, and hence $x = y$ by Property 2.1.6.6 of the Fourier transform. Thus $(x_n)_{n \geqslant 1}$ converges in E. From the factorization

$$\mu_n = (\mu_n * \varepsilon_{x_n}) * \varepsilon_{-x_n}$$

together with the continuity of the convolution in $M^1(E)$ we conclude that $(\mu_n)_{n \geqslant 1}$ τ_{w}-converges. $\qquad\square$

Corollary 2.2.14 *Let* $(\mu_n)_{n\geqslant 1}$ *be a shift tight sequence in* $M^1(E)$ *such that* $(\hat{\mu}_n)_{n\geqslant 1}$ *converges uniformly on bounded subsets of* E'. *Then* $(\mu_n)_{n\geqslant 1}$ τ_{w}-*converges.*

Proof. Theorem 2.2.12 yields the τ_{w}-relative compactness of $(\mu_n)_{n\geqslant 1}$. But then the continuity theorem 2.1.9 implies the assertion. $\qquad\square$

The next topic will be the discussion of symmetrizing measures in $M^1(E)$, which will place some of the preceding results in a more applicable setting.

Definition 2.2.15 Given measures $\mu, \nu \in M^1(E)$ we call μ a **factor** of ν if there exists $\lambda \in M^1(E)$ such that $\mu * \lambda = \nu$, in which case we write $\mu \prec \nu$.

Properties 2.2.16 of the factorization.

2.2.16.1 (Reflexivity) $\mu \prec \mu$ *for each* $\mu \in M^1(E)$.

2.2.16.2 (Weak symmetry) $\mu \prec \nu$ *and* $\nu \prec \mu$ *for* $\mu, \nu \in M^1(E)$ *implies* $\mu = \nu * \varepsilon_x$ *for some* $x \in E$.

2.2.16.3 (Transitivity) $\mu \prec \nu$ *and* $\nu \prec \kappa$ *for* $\mu, \nu, \kappa \in M^1(E)$ *implies* $\mu \prec \kappa$.

2.2.16.4 (Permanence under convolution) *If* $\mu_1 \prec \nu_1$ *and* $\mu_2 \prec \nu_2$ *for* $\mu_1, \nu_1, \mu_2, \nu_2 \in M^1(E)$ *then* $\mu_1 * \mu_2 \prec \nu_1 * \nu_2$.

2.2.16.5 *Let* $(\mu_n)_{n\geqslant 1}$ *and* $(\nu_n)_{n\geqslant 1}$ *be sequences in* $M^1(E)$ *such that* $(\nu_n)_{n\geqslant 1}$ *is shift tight and* $\mu_n \prec \nu_n$ *for all* $n \in \mathbb{N}$. *Then* $(\mu_n)_{n\geqslant 1}$ *is also shift tight.*

2.2.16.6 (Continuity of $\prec$) *Let* $(\mu_n)_{n\geqslant 1}$ *and* $(\nu_n)_{n\geqslant 1}$ *be sequences in* $M^1(E)$ *such that* $\mu_n \prec \nu_n$ *for all* $n \in \mathbb{N}$ *and both*

$$\tau_{\mathrm{w}} - \lim_{n\to\infty} \mu_n = \mu, \ \ \tau_{\mathrm{w}} - \lim_{n\to\infty} \nu_n = \nu\,.$$

Then $\mu \prec \nu$.

2.2.16.7 *If* $\mu \prec \nu$ *for* $\mu, \nu \in M^1(E)$ *then*

$$|\hat{\nu}(a)| \leqslant |\hat{\mu}(a)|$$

$a \in E'$.

Proof. To prove 2.2.16.2, suppose $\mu * \lambda = \nu$ and $\nu * \kappa = \mu$ for $\lambda, \kappa \in M^1(E)$. Then $\mu * \lambda * \kappa = \mu$. From Lemma 2.2.2 we infer that $\lambda * \kappa = \varepsilon_0$, and hence by Corollary 1.4.7 that there exists $x \in E$ such that $\lambda = \varepsilon_{-x}$ and $\kappa = \varepsilon_x$. 2.2.16.5 follows from Theorem 2.2.7. To show

2.2.16.6, given $\mu_n * \lambda_n = \nu_n$ for $\lambda_n \in M^1(E)$, $n \in \mathbb{N}$, Theorem 2.2.3 yields the τ_{w}-relative compactness of $(\lambda_n)_{n \geqslant 1}$. Let λ be an accumulation point of $(\lambda_n)_{n \geqslant 1}$. The continuity of the convolution in $M^1(E)$ implies that $\mu * \lambda = \nu$, and this is the desired assertion.

And finally, 2.2.16.7 follows from $\hat{\mu}(a)\hat{\lambda}(a) = \hat{\nu}(a)$ for some $\lambda \in M^1(E)$ together with $|\hat{\lambda}(a)| \leqslant 1$ for all $a \in E'$. $\square$

Definition 2.2.17 For every $\mu \in M^{\mathrm{b}}(E)$ the **adjoint** μ^- of μ is defined (as a measure in $M^{\mathrm{b}}(E)$) by

$$\mu^-(B) := \mu(-B)$$

whenever $B \in \mathfrak{B}(E)$. We refer to μ as **symmetric** if $\mu^- = \mu$.

Properties 2.2.18 of the adjoint.
Let $x \in E$, $\mu, \nu \in M^{\mathrm{b}}(E)$.

2.2.18.1 $(\varepsilon_x)^- = \varepsilon_{-x}$.

2.2.18.2 $(\mu^-)^\wedge = \bar{\hat{\mu}}$.

2.2.18.3 μ is symmetric if and only if $\hat{\mu}$ is real-valued, and this in turn holds if and only if

$$\hat{\mu}(a) = \int \cos\langle x, a \rangle \, \mu(\mathrm{d}x)$$

$a \in E'$.

2.2.18.4 The mapping $\mu \mapsto \mu^-$ from $M^{\mathrm{b}}(E)$ into itself is τ_{w}-continuous.

2.2.18.5 $(\mu^-)^- = \mu$.

2.2.18.6 $(\mu * \nu)^- = \mu^- * \nu^-$.

In particular, the convolution of symmetric measures is again symmetric.

Theorem 2.2.19 For every sequence $(\mu_n)_{n \geqslant 1}$ of symmetric measures in $M^1(E)$ the following statements are equivalent:

(i) $(\mu_n)_{n \geqslant 1}$ is τ_{w}-relatively compact.
(ii) $(\mu_n)_{n \geqslant 1}$ is shift tight.

Proof. It suffices to demonstrate the implication (ii) $\Rightarrow$ (i). Let $(x_n)_{n \geqslant 1}$ be a centralizing sequence for $(\mu_n)_{n \geqslant 1}$. For every $n \in \mathbb{N}$ we have that

$$(\mu_n * \varepsilon_{x_n})^- = \mu_n^- * \varepsilon_{-x_n} = \mu_n * \varepsilon_{-x_n} \, .$$

Since the adjoint is a continuous mapping by Property 2.2.18.4, $(-x_n)_{n \geqslant 1}$ is also a centralizing sequence for $(\mu_n)_{n \geqslant 1}$. But then $(2x_n)_{n \geqslant 1}$ is relatively compact in E by Corollary 2.2.10, and Theorem 2.2.9 gives the result. $\square$

Theorem 2.2.20 *Let* $(\mu_n)_{n \geqslant 1}$ *be a sequence of symmetric measures in* $M^1(E)$ *and* $(x_n)_{n \geqslant 1}$ *any sequence in* E *such that* $(\mu_n * \varepsilon_{x_n})_{n \geqslant 1}$ τ_{w}-*converges. Then the sequences* $(\mu_n)_{n \geqslant 1}$ *and* $(x_n)_{n \geqslant 1}$ *converge in* $M^1(E)$ *and* E *respectively.*

Proof. From $\tau_{\mathrm{w}} - \lim_{n \to \infty} \mu_n * \varepsilon_{x_n} = \lambda \in M^1(E)$ it follows that

$$\tau_{\mathrm{w}} - \lim_{n \to \infty} \mu_n * \varepsilon_{-x_n} = \lambda^- .$$

On the other hand we have

$$\mu_n * \varepsilon_{x_n} = (\mu_n * \varepsilon_{-x_n}) * \varepsilon_{2x_n}$$

for all $n \in \mathbb{N}$. From Corollary 2.2.11 we infer that $\lim_{n \to \infty} 2x_n =: y \in E$ exists, and hence

$$x := \lim_{n \to \infty} x_n = \frac{1}{2} y .$$

Finally, the continuity of the convolution in $M^1(E)$ yields

$$\tau_{\mathrm{w}} - \lim_{n \to \infty} \mu_n = \tau_{\mathrm{w}} - \lim_{n \to \infty} (\mu_n * \varepsilon_{x_n}) * \varepsilon_{-x_n} = \lambda * \varepsilon_{-x} . \qquad \square$$

Theorem 2.2.21 *Let* $(\mu_n)_{n \geqslant 1}$, $(\nu_n)_{n \geqslant 1}$ *be sequences in* $M^1(E)$, *where each* μ_n *is symmetric. If* $(\mu_n * \nu_n)_{n \geqslant 1}$ *is a* τ_{w}-*relatively compact sequence, then so are* $(\mu_n)_{n \geqslant 1}$ *and* $(\nu_n)_{n \geqslant 1}$.

Proof. Theorem 2.2.7 implies that $(\mu_n)_{n \geqslant 1}$ is shift compact. Then, by Theorem 2.2.19, $(\mu_n)_{n \geqslant 1}$ is τ_{w}-relatively compact and, by Theorem 2.2.3, $(\nu_n)_{n \geqslant 1}$ is τ_{w}-relatively compact. $\qquad \square$

Theorem 2.2.22 *For every sequence* $(\mu_n)_{n \geqslant 1}$ *in* $M^{\mathrm{b}}(E)$ *the following are equivalent:*

(i) $(\mu_n)_{n \geqslant 1}$ *is* τ_{w}-*relatively compact.*
(ii) $(\mu_n + \mu_n^-)_{n \geqslant 1}$ *is* τ_{w}-*relatively compact.*

Proof. For every $n \geqslant 1$ we have

$$(\mu_n + \mu_n^-)(E) = \mu_n(E) + \mu_n^-(E) = 2\mu_n(E)$$

and hence

$$\sup_{n \geqslant 1} (\mu_n + \mu_n^-)(E) = 2 \sup_{n \geqslant 1} \mu_n(E) .$$

Let $\varepsilon > 0$ and $K \in \mathcal{K}(E)$. If $\mu_n(\complement K) < \varepsilon$ then

$$(\mu_n + \mu_n^-)(\complement(K \cup (-K))) \leqslant \mu_n(\complement K) + \mu_n^-(-\complement K)$$
$$= 2\mu_n(\complement K) < \varepsilon$$

and $K \cup (-K) \in \mathcal{K}(E)$. On the other hand, if

$$(\mu_n + \mu_n^-)(\complement K) < \varepsilon$$

we obtain that $\mu_n(\complement K) < \varepsilon$ for every $n \geqslant 1$. Prohorov's theorem 1.3.7 yields both of the desired implications. $\qquad\square$

Definition 2.2.23 For every $\mu \in M^1(E)$ the measure

$$|\mu|^2 := \mu * \mu^- \in M^1(E)$$

is called the **symmetrization** of μ.

Properties 2.2.24 of the symmetrization.
Let $\mu, \nu \in M^1(E)$.

2.2.24.1 $|\mu|^2$ is symmetric.

2.2.24.2 $|\mu * \nu|^2 = |\mu|^2 * |\nu|^2$, and in particular, $|\mu^n|^2 = (|\mu|^2)^n$ for all $n \in \mathbb{N}$.

2.2.24.3 $(|\mu|^2)^{\wedge}(a) = |\hat{\mu}(a)|^2$ for all $a \in E'$.

Proof. 2.2.24.1 is a simple application of Properties 2.2.18.5 and 2.2.18.6. To show 2.2.24.2, it suffices to apply Property 2.2.18.6, and for 2.2.24.3 we just appeal to Properties 2.1.6.7 and 2.2.18.2 (of the Fourier transform). $\square$

Theorem 2.2.25 For every sequence $(\mu_n)_{n \geqslant 1}$ in $M^1(E)$ the following are equivalent:

(i) $(\mu_n)_{n \geqslant 1}$ is shift tight.
(ii) $(|\mu_n|^2)_{n \geqslant 1}$ is τ_{w}-relatively compact.

Proof. (i) $\Rightarrow$ (ii). From the τ_{w}-continuity of the mapping $\mu \mapsto \mu^-$ (Property 2.2.18.4) we obtain that $(\mu_n^-)_{n \geqslant 1}$ is shift tight. An application of Corollary 2.2.8 yields the shift tightness of $(|\mu_n|^2)_{n \geqslant 1}$. Finally, we employ Property 2.2.24.1 together with Theorem 2.2.19 in order to see that $(|\mu_n|^2)_{n \geqslant 1}$ is τ_{w}-relatively compact.

(ii) $\Rightarrow$ (i). This follows directly from Corollary 2.2.8. $\qquad\square$

Corollary 2.2.26 For every $\mu \in M^1(E)$ the following are equivalent:

(i) $(\mu^n)_{n \geqslant 1}$ is shift tight.
(ii) μ is a Dirac measure.

Proof. It suffices to demonstrate the implication (i) $\Rightarrow$ (ii). Theorem 2.2.25 together with Property 2.2.24.2 implies that $((|\mu|^2)^n)_{n \geqslant 1}$ is τ_{w}-relatively compact. But then Lemma 2.2.2 gives $|\mu|^2 = \varepsilon_0$, and μ itself must be a Dirac measure. $\qquad\square$

2.3   Infinitely divisible and embeddable measures

We start by giving the basic

Definition 2.3.1 A measure $\mu \in M^1(E)$ is called **infinitely divisible** if for every $n \in \mathbb{N}$ there exists an n-th **root** of μ i.e. a measure $\mu_n \in M^1(E)$ such that

$$\mu_n^n = \mu \,.$$

By $I(E)$ we denote the set of all infinitely divisible measures in $M^1(E)$.

Example 2.3.2 For every $x \in E$ the Dirac measure ε_x belongs to $I(E)$.

In fact, for every $n \in \mathbb{N}$

$$\varepsilon_x = (\varepsilon_{\frac{1}{n}x})^n \,,$$

since E is divisible.

Clearly, $I(E)$ is a (commutative) subsemigroup of $M^1(E)$. One just notes that for $\mu, \nu \in I(E)$ admitting the representations $\mu = \mu_n^n$ and $\nu = \nu_n^n$ with $\mu_n, \nu_n \in M^1(E)$ respectively $(n \in \mathbb{N})$, we have

$$\mu * \nu = (\mu_n * \nu_n)^n \,.$$

Moreover, the semigroup $I(E)$ has a neutral element ε_0.

Theorem 2.3.3 *For each $\mu \in I(E)$ one has $\hat{\mu}(a) \neq 0$ whenever $a \in E'$.*

Proof.   Let $\mu = \mu_n^n$ with $\mu_n \in M^1(E)$ for every $n \in \mathbb{N}$. Then $\mu_n \prec \mu$ for all $n \in \mathbb{N}$, and by Property 2.2.16.5 $(\mu_n)_{n \geqslant 1}$ is shift tight, hence by Theorem 2.2.25 $(|\mu_n|^2)_{n \geqslant 1}$ is τ_{w}-relatively compact. Let ν be an accumulation point of $(|\mu_n|^2)_{n \geqslant 1}$. Then ν is symmetric. Moreover, for all $n \geqslant m$ we have that

$$(|\mu_n|^2)^m \prec |\mu|^2 \,,$$

and consequently that

$$\nu^m \prec |\mu|^2$$

for all $m \in \mathbb{N}$, since $\prec$ is a closed relation. But then $(\nu^m)_{m \geqslant 1}$ is a τ_{w}-relatively compact sequence, which follows with the help of Property 2.2.16.5 and Theorem 2.2.19. Thus $\nu = \varepsilon_0$ which implies that

$$|\mu_n|^2 \to \varepsilon_0$$

as $n \to \infty$ (in the topology τ_w) and therefore by the continuity theorem 2.1.9 together with Property 2.2.24.3 that

$$\lim_{n \to \infty} |\hat{\mu}_n(a)|^2 = 1$$

for all $a \in E'$. But then for each $a \in E'$ there exists an $n \in \mathbb{N}$ such that $\hat{\mu}_n(a) \neq 0$, thus

$$\hat{\mu}(a) = (\hat{\mu}_n(a))^n \neq 0 .$$

$\square$

Corollary 2.3.4 *For every* $\mu \in I(E)$

(i) *there exists* $\operatorname{Log} \hat{\mu}$, *and*

(ii) *if* μ *is symmetric, then* $\operatorname{Log} \hat{\mu}$ *is real which implies that* $\hat{\mu}(a) > 0$ *whenever* $a \in E'$.

Proof. (i) follows from Theorem 2.1.10 on the existence of the function h. Concerning

(ii) we first note that $\hat{\mu}(E') \subset \mathbb{R}$, since μ is symmetric and hence $\hat{\mu}$ is real. Next we realize that

$$\hat{\mu}(a) = \exp \operatorname{Log} \hat{\mu}(a)$$

for all $a \in E'$, hence that

$$(\operatorname{Log} \hat{\mu})(E') \subset \mathbb{R} \cup 2\pi i \mathbb{Z}^{\times} .$$

As a continuous image of E' the set $(\operatorname{Log} \hat{\mu})(E')$ is connected. But $\operatorname{Log} \hat{\mu}(0) = 0$, thus $(\operatorname{Log} \hat{\mu})(E') \subset \mathbb{R}$ and the remaining assertion follows.

$\square$

Theorem 2.3.5 (Uniqueness of roots)
Let $\mu \in I(E)$. *For every* $n \in \mathbb{N}$ *there exists exactly one (n-th root)* $\mu_n \in M^1(E)$ *such that* $\mu_n^n = \mu$, *and*

$$\hat{\mu}_n(a) = \exp \left(\frac{1}{n} \operatorname{Log} \hat{\mu}(a) \right)$$

whenever $a \in E'$.

If, moreover, μ *is symmetric, then also each n-th root* μ_n *of* μ *is symmetric* $(n \in \mathbb{N})$.

Proof. From Theorem 2.3.3 we conclude that $\hat{\mu}(a) \neq 0$ for all $a \in E'$. But then also $\hat{\mu}_n(a) \neq 0$ for each $n \in \mathbb{N}$ $(a \in E')$. Theorem 2.1.10 yields the existence of $\operatorname{Log} \hat{\mu}_n$ for every $n \in \mathbb{N}$. We also have that

$$\exp(\operatorname{Log} \hat{\mu}(a)) = \hat{\mu}(a) = (\hat{\mu}_n(a))^n = \exp(n \operatorname{Log} \hat{\mu}_n(a))$$

for each $n \in \mathbb{N}$ $(a \in E')$, hence that

$$\text{Log}\,\hat{\mu} = n \,\text{Log}\,\hat{\mu}_n$$

or

$$\text{Log}\,\hat{\mu}_n = \frac{1}{n}\,\text{Log}\,\hat{\mu}\,.$$

The uniqueness of the Fourier transform (Theorem 2.1.4) implies the uniqueness of the n-th root μ_n of μ as asserted.

Finally, Corollary 2.3.4 together with Property 2.2.18.3 yields the last assertion of the theorem. $\qquad\square$

Theorem 2.3.6 (Convergence of roots)
For $\mu \in \mathrm{I}(E)$ with sequence $(\mu_{\frac{1}{n}})_{n \geqslant 1}$ of n-th roots $\mu_{\frac{1}{n}}$ of μ,

$$\mu_{\frac{1}{n}} \to \varepsilon_0$$

as $n \to \infty$ in the weak topology τ_{w}.

Proof. Let $\delta > 0$. By Corollary 2.1.12

$$\sup\{|\text{Log}\,\hat{\mu}(a)| : a \in V_\delta\} < \infty\,.$$

From Theorem 2.3.5 we infer that

$$\lim_{n\to\infty} (\mu_{\frac{1}{n}})^{\wedge}(a) = \lim_{n\to\infty} \exp\left(\frac{1}{n}\,\text{Log}\,\hat{\mu}(a)\right)$$
$$= 1$$
$$= \hat{\varepsilon}_0(a)$$

uniformly in $a \in V_\delta$. Since $\mu_{\frac{1}{n}} \prec \mu$ for all $n \in \mathbb{N}$, the sequence $(\mu_{\frac{1}{n}})_{n \geqslant 1}$ is shift tight by Property 2.2.16.5. But then Corollary 2.2.14 applies, and with the help of the uniqueness and continuity theorems 2.1.4 and 2.1.9 respectively the result follows. $\qquad\square$

Theorem 2.3.7 (Closedness of $\mathrm{I}(E)$ in $M^1(E)$)
Let $(\mu_k)_{k \geqslant 1}$ be a sequence of measures in $\mathrm{I}(E)$ such that

$$\mu_k \to \mu \in M^1(E)$$

as $k \to \infty$, with respect to τ_{w}. Then $\mu \in \mathrm{I}(E)$, and

$$(\mu_k)_{\frac{1}{n}} \to \mu_{\frac{1}{n}}$$

as $k \to \infty$, with respect to τ_{w}, for all $n \in \mathbb{N}$.

Proof. 1. Let $n \in \mathbb{N}$ be fixed. Then, by the continuity of the mappings $\mu \mapsto \mu^-$ and $(\mu, \nu) \mapsto \mu * \nu$ we obtain

$$|(\mu_k)_{\frac{1}{n}}|^2 \prec |\mu_k|^2$$

and

$$|\mu_k|^2 \to |\mu|^2$$

as $k \to \infty$, with respect to τ_w. But then Property 2.2.16.5 and Theorem 2.2.19 apply and yield the τ_w-relative compactness of $(|(\mu_k)_{\frac{1}{n}}|^2)_{k \geqslant 1}$.

Moreover, Theorem 2.3.5 together with the continuity theorem 2.1.9 implies that for all $a \in E'$

$$\lim_{k \to \infty} (|(\mu_k)_{\frac{1}{n}}|^2)^\wedge(a) = \lim_{k \to \infty} |((\mu_k)_{\frac{1}{n}})^\wedge(a)|^2$$
$$= \lim_{k \to \infty} |\hat{\mu}_k(a)|^{2/n}$$
$$= |\hat{\mu}(a)|^{2/n}.$$

Applying the continuity theorem 2.1.9 again there exists a measure $\nu_n \in M^1(E)$ satisfying

$$\hat{\nu}_n(a) = |\hat{\mu}(a)|^{2/n}$$

Since this equality holds for every $n \in \mathbb{N}$, $|\mu|^2 \in I(E)$, hence by Theorem 2.3.3

$$|\hat{\mu}(a)|^2 = (|\mu|^2)^\wedge(a) \neq 0$$

whenever $a \in E'$.

2. From 1. and from the existence of the function h (Theorem 2.1.10) we infer that $\mathrm{Log}\,\hat{\mu}$ exists. Then Corollary 2.1.16 implies that

$$\lim_{k \to \infty} \mathrm{Log}\,\hat{\mu}_k(a) = \mathrm{Log}\,\hat{\mu}(a)$$

uniformly on bounded subsets of elements a of E'. It follows from Theorem 2.3.5 that

$$\lim_{k \to \infty} ((\mu_k)_{\frac{1}{n}})^\wedge(a) = \lim_{k \to \infty} \exp\left(\frac{1}{n} \mathrm{Log}\,\hat{\mu}_k(a)\right)$$
$$= \exp\left(\frac{1}{n} \mathrm{Log}\,\hat{\mu}(a)\right)$$

uniformly on bounded subsets of elements a in E'. In addition, $((\mu_k)_{\frac{1}{n}})_{k \geqslant 1}$ is a shift tight sequence (by Property 2.2.16.5), consequently, by Corollary 2.2.14 there exists a measure $\mu_n \in M^1(E)$ such that

$$(\mu_k)_{\frac{1}{n}} \to \mu_n$$

for all $n \in \mathbb{N}$ ($k \to \infty$, with respect to τ_w). But then the continuity of the convolution yields $\mu_n^n = \mu$ or $\mu \in I(E)$. By Theorem 2.3.5 we have that $\mu_n = \mu_{\frac{1}{n}}$ for all $n \in \mathbb{N}$. $\qquad\square$

Definition 2.3.8 A family $(\mu_t)_{t \in \mathbb{R}_+}$ of measures in $M^1(E)$ is said to be a (continuous) **convolution semigroup** in $M^1(E)$ if it has the following properties:

(a) $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \in \mathbb{R}_+$.
(b) $\mu_0 = \varepsilon_0$.
(c) The mapping $t \mapsto \mu_t$ from $\mathbb{R}_+$ into $M^1(E)$ is τ_{w}-continuous.

Evidently, the measures μ_t of a convolution semigroup $(\mu_t)_{t \in \mathbb{R}_+}$ in $M^1(E)$ are elements of $I(E)$, since

$$\mu_t = (\mu_{\frac{t}{n}})^n$$

for every $n \in \mathbb{N}$. For the converse we prove the subsequent

Theorem 2.3.9 (Embedding)
Let $\mu \in I(E)$. Then for every $t \in \mathbb{R}_+$ there exists exactly one measure $\mu_t \in M^1(E)$ such that

$$\hat{\mu}_t(a) = \exp(t \operatorname{Log} \hat{\mu}(a))$$

whenever $a \in E'$, and $(\mu_t)_{t \in \mathbb{R}_+}$ is a convolution semigroup in $M^1(E)$.
    In particular one has

$$\mu_1 = \mu.$$

Proof.   1. Let $t \in \mathbb{Q}_+^{\times}$ and $k, \ell, k', \ell' \in \mathbb{N}$ such that

$$\frac{k}{\ell} = t = \frac{k'}{\ell'}$$

By Theorem 2.3.5 we obtain that for all $a \in E'$

$$\left((\mu_{\frac{1}{\ell}})^k\right)^{\wedge}(a) = \left((\mu_{\frac{1}{\ell}})^{\wedge}(a)\right)^k = \left(\exp\frac{1}{\ell}\operatorname{Log}\hat{\mu}(a)\right)^k$$

$$= \exp\left(\frac{k}{\ell}\operatorname{Log}\hat{\mu}(a)\right) = \exp(t\operatorname{Log}\hat{\mu}(a))$$

$$= \left((\mu_{\frac{1}{\ell'}})^{k'}\right)^{\wedge}(a).$$

The uniqueness theorem 2.1.4 then implies that

$$\mu_t = (\mu_{\frac{1}{\ell}})^k$$

is uniquely determined, and

$$\mu_s * \mu_t = \mu_{s+t}$$

whenever $s, t \in \mathbb{Q}_+^\times$.

2. Let $t \in \mathbb{R}_+$ and let $(t_n)_{n \geqslant 1}$ be a sequence in $\mathbb{Q}_+^\times$ such that $\lim_{n \to \infty} t_n = t$. Let, moreover, $m \in \mathbb{N}$ with $t_n < m$ for all $n \in \mathbb{N}$.

Since by 1.

$$\mu_{t_n} * \mu_{m - t_n} = \mu_m$$

for all $n \in \mathbb{N}$, $(\mu_{t_n})_{n \geqslant 1}$ is shift tight. This follows from Theorem 2.2.7.

Now, let $\delta > 0$ be given. We know that

$$\sup\{|\operatorname{Log} \hat{\mu}(a)| : a \in V_\delta\} < \infty.$$

But then 1. implies that

$$\lim_{n \to \infty} \hat{\mu}_{t_n}(a) = \lim_{n \to \infty} \exp(t_n \operatorname{Log} \hat{\mu}(a))$$
$$= \exp(t \operatorname{Log} \hat{\mu}(a))$$

uniformly for $a \in V_\delta$. It follows from Corollary 2.2.14 and from the uniqueness and continuity properties of the Fourier transform that there exists exactly one measure $\mu_t \in M^1(E)$ satisfying

$$\hat{\mu}_t(a) = \exp(t \operatorname{Log} \hat{\mu}(a))$$

for all $a \in E'$. Moreover,

$$\lim_{n \to \infty} \mu_{t_n} = \mu_t$$

which shows the first assertion in the theorem.

3. As a consequence of the previous discussion we obtain that

$$\mu_s * \mu_t = \mu_{s+t}$$

for all $s, t \in \mathbb{R}_+$. The continuity of $t \mapsto \mu_t$ follows as in 2. by choosing for $t \in \mathbb{R}_+$ an arbitrary sequence $(t_n)_{n \geqslant 1}$ in $\mathbb{R}_+^\times$ such that $\lim_{n \to \infty} t_n = t$. □

2.4 Gauss and Poisson measures

In this section the prominent subsets of Gauss and Poisson measures within $I(E)$ will be studied.

Definition 2.4.1 A measure $\rho \in M^1(E)$ is called a (symmetric) **Gauss measure** if there exists a symmetric linear mapping $R : E' \to E$ such that

$$\hat{\rho}(a) = \exp\left(-\frac{1}{2}\langle Ra, a \rangle\right)$$

whenever $a \in E'$.

Here we apply the symmetry of a mapping $R : E' \to E$ in the sense that

$$\langle Ra, b \rangle = \langle a, Rb \rangle$$

for all $a, b \in E'$.

R is said to be the **covariance operator** of ρ.

By $G(E)$ we abbreviate the totality of all Gauss measures on E.

Examples 2.4.2

2.4.2.1 Let $E := \mathbb{R}^p$. Then the p-**dimensional normal distribution** $N(0, C)$ with mean (vector) 0 and covariance (matrix) C is a Gauss measure on E.

In fact, for all $a \in \mathbb{R}^p = (\mathbb{R}^p)'$ we have that

$$N(0, C)^{\wedge}(a) = \exp\left(-\frac{1}{2}\langle Ca, a \rangle\right),$$

and C is symmetric.

2.4.2.2 Let $E := C(I)$ (equal to the space of continuous real-valued functions on the compact interval $I := [0, 1]$). Then the **Wiener measure** W is a Gauss measure on E.

In fact, from the theory of Brownian motion follows that

$$\widehat{W}(\chi) = \exp\left(-\frac{1}{2}\langle R\chi, \chi \rangle\right)$$

with

$$(R\chi)(t) := \int (s \wedge t)\,\chi(\mathrm{d}s)$$

for all $\chi \in C(I)' \cong M^{\mathrm{b}}(I) - M^{\mathrm{b}}(I)$, $t \in I$. It turns out that R is a linear mapping $C(I)' \to C(I)$ which is shown to be symmetric (by Fubini's theorem).

Scholium 2.4.3 on the Wiener measure.

Let $(\Omega, \mathfrak{A}, \mathsf{P}, (B_t)_{t\in I})$ denote a Brownian motion in $\mathbb{R}$ with parameter set $I = [0,1]$. For every $t \in I$ there exists a real-valued random variable $\widetilde{B}_t$ on $(\Omega, \mathfrak{A}, \mathsf{P})$ with $\mathsf{P}([\widetilde{B}_t \neq B_t]) = 0$ such that for every $\omega \in \Omega$ the mapping $t \mapsto \widetilde{B}_t(\omega)$ from I into $\mathbb{R}$ is continuous. The process $(\Omega, \mathfrak{A}, \mathsf{P}, (\widetilde{B}_t)_{t\in I})$ is called a Brownian motion with continuous paths. As a result of this modification we may assume without loss of generality that the initial Brownian motion $(\Omega, \mathfrak{A}, \mathsf{P}, (B_t)_{t\in I})$ has continuous paths. It follows that the mapping

$$(\omega, t) \mapsto B_t(\omega)$$

from $\Omega \times I$ into $\mathbb{R}$ is $\mathfrak{A} \otimes \mathfrak{B}(I)$–$\mathfrak{B}(\mathbb{R})$-measurable, hence that the mapping $B : \Omega \to C(I)$ which sends ω onto the path $t \mapsto B_t(\omega)$ is $\mathfrak{A}$–$\mathfrak{B}(C(I))$-measurable. But then the Wiener measure $W := \mathsf{P}_B = B(\mathsf{P})$ is a measure in $M^1(C(I))$ with Fourier transform given by

$$\widehat{W}(\chi) = \exp\left(-\frac{1}{2}\int (s \wedge t)\,\chi \otimes \chi(\mathrm{d}s, \mathrm{d}t)\right)$$

for all $\chi \in C(I)'$.

Now let

$$(R\chi)(t) := \int (s \wedge t)\,\chi(\mathrm{d}s)$$

for all $\chi \in C(I)'$, $t \in I$. Since

$$|(R\chi)(t_1) - (R\chi)(t_2)| \leqslant |t_1 - t_2| \|\chi\|$$

for all $\chi \in C(I)'$, $t_1, t_2 \in I$, $R\chi$ is Lipschitz continuous, in particular, $R\chi \in C(I)$. Clearly, R is a continuous linear mapping $C(I)' \to C(I)$. Moreover,

$$\int (s \wedge t)\chi \otimes \chi(\mathrm{d}s, \mathrm{d}t) = \int \left(\int (s \wedge t)\,\chi(\mathrm{d}s)\right)\chi(\mathrm{d}t)$$
$$= \int (R\chi)(t)\,\chi(\mathrm{d}t) = \langle R\chi, \chi \rangle$$

and therefore

$$\widehat{W} = \exp\left(-\frac{1}{2}\langle R\chi, \chi \rangle\right)$$

whenever $\chi \in C(I)'$.

Properties 2.4.4 of Gauss measures.

2.4.4.1 *R is uniquely determined and positive in the sense that*

$$\langle Ra, a \rangle \geqslant 0$$

for all $a \in E'$.

In fact, the uniqueness of R follows from the representation

$$\langle Ra, a \rangle = - \operatorname{Log} \hat{\rho}(a)$$

together with

$$\langle Ra, b \rangle = \frac{1}{4} \langle R(a+b), a+b \rangle - \frac{1}{4} \langle R(a-b), a-b \rangle \,,$$

both equalities being valid for all $a, b \in E'$.

The positivity of R is a consequence of

$$\hat{\rho}(a) \leqslant \hat{\rho}(0) = 1$$

valid for all $a \in E'$.

2.4.4.2 *Every Gauss measure is symmetric, since $\hat{\rho}$ is real-valued.*

2.4.4.3 *Let $\rho, \sigma \in \mathrm{G}(E)$ with covariance operators R and S respectively. Then $\rho * \sigma \in \mathrm{G}(E)$ with covariance operator $R+S$. In particular, $\mathrm{G}(E)$ is a subsemigroup of $M^1(E)$.*

As for the proof of this property we just compute

$$\begin{aligned}
(\rho * \sigma)^{\wedge}(a) &= \hat{\rho}(a)\hat{\sigma}(a) \\
&= \exp\left(-\frac{1}{2}\langle Ra, a \rangle\right) \exp\left(-\frac{1}{2}\langle Sa, a \rangle\right) \\
&= \exp\left(-\frac{1}{2}\big(\langle Ra, a \rangle + \langle Sa, a \rangle\big)\right) \\
&= \exp\left(-\frac{1}{2}\langle (R+S)a, a \rangle\right)
\end{aligned}$$

for all $a \in E'$ and observe that $R + S$ is a symmetric linear mapping $E' \to E$.

2.4.4.4 *Let $\rho \in \mathrm{G}(E)$ with covariance operator R and let T be a continuous linear mapping from E into another Banach space F. Then $T(\rho) \in \mathrm{G}(F)$ with covariance operator $T \circ R \circ T^{\top}$.*

In fact, for all $b \in F'$ we have

$$\begin{aligned}
T(\rho)^{\wedge}(b) &= \hat{\rho} \circ T^{\top}(b) = \hat{\rho}(T^{\top}b) = \exp\left(-\frac{1}{2}\langle R(T^{\top}b), T^{\top}b \rangle\right) \\
&= \exp\left(-\frac{1}{2}\langle (T \circ R \circ T^{\top})b, b \rangle\right),
\end{aligned}$$

and $T \circ R \circ T^\top$ is a symmetric mapping $F' \to F$.

2.4.4.5 $G(E) \subset I(E)$.

Moreover, for every $\rho \in G(E)$ there exists a convolution semigroup $(\rho_t)_{t \in \mathbb{R}_+}$ given by

$$\rho_t := H_{\sqrt{t}}(\rho)$$

for all $t \in \mathbb{R}_+$ such that $\rho_1 = \rho$.

Here the symbol H_r denotes the homothetical mapping $x \mapsto rx$ (with $r \in \mathbb{R}_+$) on E.

For the proof of this property we first agree on R to be the covariance operator of ρ. Then by Property 2.4.4.4 $H_{\sqrt{t}}(\rho) \in G(E)$ with covariance operator tR. It follows that

$$\left(H_{\sqrt{\frac{1}{n}}}(\rho) \right)^n = \rho$$

for every $n \in \mathbb{N}$, hence that $\rho \in I(E)$. The remaining part of the statement follows from the embedding theorem 2.3.9.

2.4.4.6 *Let $s, t \in \mathbb{R}$ with $s^2 + t^2 = 1$, and let $T := T_{s,t}$ be defined by*

$$T(x, y) := (sx + ty, tx - sy)$$

for all $(x, y) \in E \times E$. T is a continuous linear mapping on $E \times E$. Then for every $\rho \in G(E)$

$$T(\rho \otimes \rho) = \rho \otimes \rho.$$

In fact, let R denote the covariance operator of ρ, and let

$$q(a) := \langle Ra, a \rangle$$

for all $a \in E'$. Then

$$q(sa + tb) + q(ta - sb) = q(a) + q(b)$$

and

$$T^\top(a, b) = (sa + tb, ta - sb)$$

whenever $a, b \in E'$. Here the identification $(E \times E)' = E' \times E'$ is applied.

It follows that for all $(a, b) \in E' \times E'$

$$
\begin{aligned}
(T(\rho \otimes \rho))^\wedge(a, b) &= (\rho \otimes \rho)^\wedge(T^\top(a, b)) = \hat{\rho}(sa + tb)\hat{\rho}(ta - sb) \\
&= \exp\left(-\frac{1}{2}\big[q(sa + tb) + q(ta - sb)\big]\right) \\
&= \exp\left(-\frac{1}{2}\big[q(a) + q(b)\big]\right) \\
&= \hat{\rho}(a)\hat{\rho}(b) = (\rho \otimes \rho)^\wedge(a, b).
\end{aligned}
$$

Lemma 2.4.5 (Fernique)
For every $\rho \in \mathrm{G}(E)$

$$\int \|x\|^2 \, \rho(\mathrm{d}x) < \infty \, .$$

Proof. 1. The mapping $T := T_{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}}$ defined as in Property 2.4.4.6 is invertible with $T^{-1} = T$. Now, let $x, y \in E$, $u, v \in \mathbb{R}_+$ with $\|x\| \leqslant u$ and $\|y\| > v$. Then

$$\left\| \frac{1}{\sqrt{2}}(x \pm y) \right\| \geqslant \frac{1}{\sqrt{2}}(\|y\| - \|x\|) > \frac{v - u}{\sqrt{2}} \, ,$$

and by Property 2.4.4.6 we obtain

$$
\begin{aligned}
\rho([\|\cdot\| \leqslant u])\rho([\|\cdot\| > v]) &= \rho \otimes \rho([\|\cdot\| \leqslant u] \times [\|\cdot\| > v]) \\
&= T(\rho \otimes \rho)([\|\cdot\| \leqslant u] \times [\|\cdot\| > v]) \\
&= \rho \otimes \rho(T([\|\cdot\| \leqslant u] \times [\|\cdot\| > v])) \\
&\leqslant \rho \otimes \rho\left(\left[\|\cdot\| > \frac{v - u}{\sqrt{2}}\right] \times \left[\|\cdot\| > \frac{v - u}{\sqrt{2}}\right] \right) \\
&= \left(\rho\left(\left[\|\cdot\| > \frac{v - u}{\sqrt{2}}\right] \right) \right)^2 .
\end{aligned}
$$

2. Now we choose $v_0 := u$ sufficiently large such that

$$\alpha := \rho([\|\cdot\| \leqslant v_0]) > \frac{3}{4} \, ,$$

and we obtain

$$\rho([\|\cdot\| > v_0]) < \frac{1}{4} \, .$$

Moreover, let

$$v_n := \left(2^{\frac{n+1}{2}} - 1 \right)\left(\sqrt{2} + 1 \right) v_0$$

for all $n \in \mathbb{N}$. Then

$$v_{n+1} - v_0 = \sqrt{2} \, v_n$$

for all $n \in \mathbb{Z}_+$. The estimate achieved in 1. yields

$$\alpha \rho([\|\cdot\| > v_{n+1}]) \leqslant (\rho([\|\cdot\| > v_n]))^2$$

for all $n \in \mathbb{Z}_+$, hence by induction

$$\frac{1}{\alpha}\rho([\|\cdot\| > v_n]) \leqslant \left(\frac{1}{\alpha}\rho([\|\cdot\| > v_0]) \right)^{2^{n-1}} < \left(\frac{1}{3} \right)^{2^{n-1}} \, .$$

3. It remains to be shown that

$$\int\limits_{[\|\cdot\|>v_1]} \|x\|^2 \rho(dx) < \infty .$$

In fact, from $v_n \leqslant 4v_0 2^{\frac{n}{2}}$ and from the estimate in 2. we conclude that

$$\int\limits_{[\|\cdot\|>v_1]} \|x\|^2 \rho(dx) = \sum_{n\geqslant 1} \int\limits_{[v_n<\|\cdot\|\leqslant v_{n+1}]} \|x\|^2 \rho(dx)$$

$$\leqslant \sum_{n\geqslant 1} v_{n+1}^2 \rho([\|\cdot\| > v_n])$$

$$\leqslant 16v_0^2 \sum_{n\geqslant 1} 2^{n+1} \left(\frac{1}{3}\right)^{2^{n-1}} < \infty .$$

$\square$

With this preparation we can approach the more profound properties of Gauss measures on a (separable) Banach space.

Theorem 2.4.6 (Integral representation of the covariance operator)
Let $\rho \in G(E)$ with covariance operator R. Then

(i) $Ra = \int \langle x, a\rangle x\, \rho(dx)$ *for all $a \in E'$, where the integral is understood in the sense of Bochner.*

(ii) R *is a compact operator.*

Proof. (i). At first we observe that

$$\|\langle x, a\rangle x\| \leqslant \|a\|\, \|x\|^2$$

for all $x \in E$, $a \in E'$. Therefore Lemma 2.4.5 implies that the mapping

$$x \mapsto \langle x, a\rangle x$$

is Bochner-integrable with respect to ρ.

We add a few remarks on *Bochner integrability*: Let $(\Omega, \mathfrak{A}, \mu)$ be a measure space, and let $f : \Omega \to E$ be a mapping from Ω into a separable Banach space B. f is said to be Bochner integrable if

(a) f is strongly measurable in the sense that there exist step functions $f_n : \Omega \to E$ such that

$$f_n \to f$$

as $n \to \infty$ strongly μ-almost everywhere, and such that

(b)

$$\int \|f_n(\omega) - f(\omega)\| \, \mu(\mathrm{d}\omega) \to 0$$

as $n \to \infty$.

In this case

$$\int f(\omega) \, \mu(\mathrm{d}\omega) := \lim_{n \to \infty} \int f_n(\omega) \, \mu(\mathrm{d}\omega) \,,$$

with the limit taken in the strong sense, is called the *Bochner integral of* f *with respect to* μ.

Moreover one has the useful criterion that a strongly measurable function $f : \Omega \to E$ is Bochner integrable with respect to μ if and only if $\|f\|$ is μ-integrable (in the usual sense).

We proceed with the proof of the theorem. Let

$$Sa := \int \langle x, a \rangle x \, \rho(\mathrm{d}x)$$

for all $a \in E'$. S defines a linear mapping $E' \to E$. Moreover, for every $b \in E'$ we have

$$\langle Sa, b \rangle = \int \langle x, a \rangle \langle x, b \rangle \, \rho(\mathrm{d}x) \,.$$

This representation shows that S is symmetric. Now, for all $a \in E'$

$$a(\rho)^{\wedge}(t) = \exp\left(-\frac{t^2}{2} \langle Ra, a \rangle\right) = N\big(0, \langle Ra, a \rangle\big)^{\wedge}(t)$$

whenever $t \in \mathbb{R}$, hence by Property 2.4.4.4 together with the uniqueness of the Fourier transform

$$a(\rho) = N\big(0, \langle Ra, a \rangle\big) \,.$$

But

$$\langle Ra, a \rangle = \int t^2 N(0, \langle Ra, a \rangle)(\mathrm{d}t)$$

$$= \int t^2 a(\rho)(\mathrm{d}t) = \int \langle x, a \rangle^2 \rho(\mathrm{d}x) = \langle Sa, a \rangle$$

for all $a \in E'$. Since R and S are symmetric, we obtain as in the proof of Property 2.4.4.1 that $R = S$.

(ii). Let $(a_n)_{n \geqslant 1}$ be a bounded sequence in E'. Then there exists a $\delta > 0$ such that $\|a_n\| < \delta$ for all $n \in \mathbb{N}$.

We know from Appendix B.14 that V_δ is compact and metrizable with respect to the topology $\sigma(E', E)$. Thus there exist a subsequence $(a_{n_k})_{k \geqslant 1}$ of $(a_n)_{n \geqslant 1}$ and an $a \in V_\delta$ such that

$$\lim_{k \to \infty} \langle x, a_{n_k} \rangle = \langle x, a \rangle$$

whenever $x \in E$. Applying (i) we now get

$$\| Ra_{n_k} - Ra \| \leqslant \int |\langle x, a_{nk} \rangle - \langle x, a \rangle| \, \|x\| \rho(\mathrm{d}x) \, .$$

In addition we have that

$$|\langle x, a_{n_k} \rangle - \langle x, a \rangle| \, \|x\| \leqslant 2\delta \|x\|^2$$

for all $x \in E$. By Lebesgue's dominated convergence theorem follows

$$\lim_{k \to \infty} Ra_{n_k} = Ra \, ,$$

and this finishes the proof that R is compact. $\qquad\square$

Theorem 2.4.7 (Characterization of Gauss measures)
For every measure $\rho \in M^1(E)$ the following statements are equivalent:

(i) $\rho \in \mathrm{G}(E)$.
(ii) *There exists a mapping $q : E' \to \mathbb{R}_+$ with*

$$q(ta) = t^2 q(a)$$

for all $t \in \mathbb{R}$, $a \in E'$ such that

$$\hat{\rho}(a) = \mathrm{e}^{-q(a)}$$

whenever $a \in E'$.
(iii) $a(\rho) \in \mathrm{G}(\mathbb{R})$ *for all $a \in E'$.*

Proof. (i) $\Rightarrow$ (ii). Let $\rho \in \mathrm{G}(E)$ with covariance operator R. Putting

$$q(a) = \frac{1}{2} \langle Ra, a \rangle$$

for all $a \in E'$ we immediately arrive at the assertion.
(ii) $\Rightarrow$ (iii). Let $a \in E'$. Then for all $t \in \mathbb{R}$

$$a(\rho)^\wedge(t) = \hat{\rho}(ta) = \mathrm{e}^{-q(ta)} = \mathrm{e}^{-t^2 q(a)} = N(0, 2q(a))^\wedge(t) \, ,$$

hence

$$a(\rho) = N(0, 2q(a))$$

or $a(\rho) \in \mathrm{G}(\mathbb{R})$.

(iii) $\Rightarrow$ (i). Let $a(\rho) = N(0, h(a))$ with a function $h : E' \to \mathbb{R}$. Then for all $a \in E'$

$$h(a) = \int t^2 N(0, h(a))(\mathrm{d}t) = \int t^2 a(\rho)(\mathrm{d}t) = \int \langle x, a \rangle^2 \rho(\mathrm{d}x) \qquad (1)$$

and

$$\hat{\rho}(a) = a(\rho)^\wedge(1) = N(0, h(a))^\wedge(1) = \exp\left(-\frac{1}{2}h(a)\right). \qquad (2)$$

Now, for all $x \in E$, $a \in E'$ we define

$$(Ta)(x) := \langle x, a \rangle.$$

By (1) $Ta \in L^2(E, \rho)$ for all $a \in E'$, T is a linear mapping $E' \to L^2(E, \rho)$, and by (1) and (2)

$$\exp\left(-\frac{1}{2}\|Ta\|^2\right) = \hat{\rho}(a)$$

whenever $a \in E'$.

Now, by Property 2.1.6.4 of the Fourier transform T is a continuous mapping

$$(E', \tau(E', E)) \to (L^2(E, \rho), \|\cdot\|).$$

Appendix B.13 (Arens, Mackey) provides us with the identification

$$(E', \tau(E', E))' \leftrightarrow E.$$

Consequently, $T^\top$ can be regarded as a mapping $L^2(E, \rho) \to E$ and hence $R := T^\top \circ T$ is a linear mapping $E' \to E$ with

$$\langle Ra, b \rangle = \langle T^\top(Ta), b \rangle = \langle Ta, Tb \rangle$$

for all $a, b \in E'$. This shows that R is symmetric, and setting $a = b$ one obtains that

$$\hat{\rho}(a) = \exp\left(-\frac{1}{2}\|Ta\|^2\right) = \exp\left(-\frac{1}{2}\langle Ra, a \rangle\right)$$

for all $a \in E'$, and ρ has been shown to belong to $\mathrm{G}(E)$. $\qquad \square$

Definition 2.4.8 For any given measure $\lambda \in M^{\mathrm{b}}(E)$ the measure

$$\mathrm{e}(\lambda) := \mathrm{e}^{-\|\lambda\|} \sum_{n \geqslant 0} \frac{\lambda^n}{n!} \in M^1(E)$$

with $\lambda^0 := \varepsilon_0$ is called the **Poisson measure with exponent** λ.

By $\mathrm{P}(E)$ we shall abbreviate the totality of all Poisson measures in E.

Discussion 2.4.9 of the genesis of Poisson measure in classical probability theory.

Here we have $E := \mathbb{R}^p$. Let $\lambda \in M^{\mathrm{b}}(E) \setminus \{0\}$. Moreover, let

(a) $(X_n)_{n \geqslant 1}$ denote a sequence of independent E-valued random variables on a probability space $(\Omega, \mathfrak{A}, \mathsf{P})$ such that
$$\mathsf{P}_{X_n} := \frac{1}{\|\lambda\|}\lambda$$
for all $n \geqslant 1$ and let

(b) N denote a $\mathbb{Z}_+$-valued random variable on $(\Omega, \mathfrak{A}, \mathsf{P})$, independent of $(X)_{n \geqslant 1}$ and such that
$$\mathsf{P}_N = \Pi(\|\lambda\|),$$
where $\Pi(\|\lambda\|)$ denotes the elementary Poisson distribution with parameter $\|\lambda\|$.

Now, for every $\omega \in \Omega$ let
$$S_N(\omega) := \sum_{k=1}^{N(\omega)} X_k(\omega)$$
(with $\sum_{k=1}^{0} X_k(\omega) := 0$). Then S_n is an E-valued random variable on $(\Omega, \mathfrak{A}, \mathsf{P})$ with
$$\mathsf{P}_{S_N} = \mathrm{e}(\lambda).$$

Properties 2.4.10 of Poisson measures.

2.4.10.1 *For every $a \in E'$*
$$\mathrm{e}(\lambda)^{\wedge}(a) = \exp(\hat{\lambda}(a) - \hat{\lambda}(0))$$
$$= \exp\Big(\int \big(\mathrm{e}^{\mathrm{i}\langle x,a\rangle} - 1\big)\lambda(\mathrm{d}x)\Big),$$
hence
$$\mathrm{Log}\,\mathrm{e}(\lambda)^{\wedge}(a) = \hat{\lambda}(a) - \hat{\lambda}(0).$$
The following computation serves as a proof of the first assertion:
$$\int \mathrm{e}^{\mathrm{i}\langle x,a\rangle}\mathrm{e}(\lambda)(\mathrm{d}x) = \mathrm{e}^{-\|\lambda\|}\sum_{n \geqslant 0}\frac{1}{n!}\int e^{i\langle x,a\rangle}\lambda^n(\mathrm{d}x)$$
$$= \mathrm{e}^{-\|\lambda\|}\sum_{n \geqslant 0}\frac{1}{n!}(\lambda^n)^{\wedge}(a)$$
$$= \mathrm{e}^{-\|\lambda\|}\sum_{n \geqslant 0}\frac{1}{n}!(\hat{\lambda}(a))^n$$
$$= \mathrm{e}^{-\hat{\lambda}(0)}\mathrm{e}^{\hat{\lambda}(a)} = \mathrm{e}^{\hat{\lambda}(a)-\hat{\lambda}(0)} \qquad (a \in E').$$

The second statement follows from the existence of $h := \mathrm{Log}\,\hat{\mu}$ (Theorem 2.1.10) together with the continuity of the Fourier transform.

2.4.10.2 *For* $\lambda_1, \lambda_2 \in M^{\mathrm{b}}(E)$ *we have*

$$\mathrm{e}(\lambda_1) * \mathrm{e}(\lambda_2) = \mathrm{e}(\lambda_1 + \lambda_2)\,.$$

In particular, $\mathrm{P}(E)$ *is a subsemigroup of* $M^1(E)$.

For the proof one just notes that

$$(\mathrm{e}(\lambda_1) * \mathrm{e}(\lambda_2))^{\wedge}(a) = \mathrm{e}(\lambda_1)^{\wedge}(a)\mathrm{e}(\lambda_2)^{\wedge}(a)$$
$$= \exp(\hat{\lambda}_1(a) - \hat{\lambda}_1(0)) \exp(\hat{\lambda}_2(a) - \hat{\lambda}_2(0))$$

(by Property 2.4.10.1)

$$= \exp((\lambda_1 + \lambda_2)^{\wedge}(a) - (\lambda_1 + \lambda_2)^{\wedge}(0))$$
$$= (\mathrm{e}(\lambda_1 + \lambda_2))^{\wedge}(a)$$

(again by Property 2.4.10.1), whenever $a \in E'$.

2.4.10.3 $\mathrm{P}(E) \subset \mathrm{I}(E)$.

Moreover, for any $\mathrm{e}(\lambda) \in \mathrm{P}(E)$ *there exists a convolution semigroup* $(\mathrm{e}(\lambda)_t)_{t \in \mathbb{R}_+}$ *given by*

$$\mathrm{e}(\lambda)_t := \mathrm{e}(t\lambda)$$

for all $t \in \mathbb{R}_+$ *such that* $\mathrm{e}(\lambda)_1 = \mathrm{e}(\lambda)$.

This follows from Properties 2.4.10.1 and 2.4.10.2 together with the embedding theorem 2.3.9.

2.4.10.4 *Let* $\lambda_1, \lambda_2 \in M^{\mathrm{b}}(E)$ *with* $\lambda_1 \leqslant \lambda_2$. *Then*

$$\mathrm{e}(\lambda_1) \prec \mathrm{e}(\lambda_2)\,.$$

In fact, looking at $\lambda_2 = \lambda_1 + (\lambda_2 - \lambda_1)$ with $\lambda_2 - \lambda_1 \in M^{\mathrm{b}}(E)$ Property 2.4.10.2 yields

$$\mathrm{e}(\lambda_2) = \mathrm{e}(\lambda_1) * \mathrm{e}(\lambda_2 - \lambda_2)$$

and hence the assertion.

2.4.10.5 *For every* $\lambda \in M^{\mathrm{b}}(E)$ *we have* $\mathrm{e}(\lambda)^- = \mathrm{e}(\lambda^-)$ *since for all* $a \in E'$

$$(\mathrm{e}(\lambda)^-)^{\wedge}(a) = \overline{\mathrm{e}(\lambda)^{\wedge}(a)} = \overline{\exp(\hat{\lambda}(a) - \hat{\lambda}(0))}$$
$$= \exp\left(\overline{\hat{\lambda}(a)} - \overline{\hat{\lambda}(0)}\right) = \exp\left(\widehat{\lambda^-}(a) - \widehat{\lambda^-}(0)\right)$$
$$= (\mathrm{e}(\lambda^-))^{\wedge}(a)\,.$$

2.4.10.6 *Let* $\lambda \in M^{\mathrm{b}}(E)$. *Then* $\mathrm{e}(\lambda + \lambda^-) = |\mathrm{e}(\lambda)|^2$, *since*

$$\mathrm{e}(\lambda + \lambda^-) = \mathrm{e}(\lambda) * \mathrm{e}(\lambda^-) = \mathrm{e}(\lambda) * \mathrm{e}(\lambda)^-$$

by Property 2.4.10.5, and furthermore

$$= |\mathrm{e}(\lambda)|^2\,.$$

Theorem 2.4.11 (Denseness of $P(E)$ in $I(E)$)
$P(E)$ *is a τ_w-dense subsemigroup of* $I(E)$.

In particular, for any $\mu \in I(E)$ with sequence $(\mu_{\frac{1}{n}})_{n \geqslant 1}$ of n-th roots $\mu_{\frac{1}{n}}$ of μ we have that

$$e(n\mu_{\frac{1}{n}}) \to \mu$$

as $n \to \infty$ *with respect to the topology* τ_w.

Proof. 1. Clearly,

$$e^{-n} \sum_{k=0}^{2n} \frac{n^k}{k!} \to 1$$

as $n \to \infty$, since

$$e^{-n} \sum_{k>2n} \frac{n^k}{k!} = \frac{e^{-n}n^{2n}}{(2n)!} \sum_{k \geqslant 1} \frac{n^k}{(2n+1)\cdots(2n+k)}$$

$$\leqslant \frac{e^{-n}n^{2n}}{(2n)!} \sum_{k \geqslant 1} 2^{-k} = \frac{e^{-n}n^{2n}}{(2n)!}$$

$$= \frac{e^{-2n}(2n)^{2n}}{(2n)!} \left(\frac{e}{4}\right)^n \leqslant \left(\frac{e}{4}\right)^n \leqslant \left(\frac{3}{4}\right)^n$$

for all $n \geqslant 1$.

2. The sequence $(e(n\mu_{\frac{1}{n}}))_{n \geqslant 1}$ is τ_w-relatively compact.
In fact, for $\varepsilon > 0$ there exists by 1. an $n_0 \in \mathbb{N}$ such that

$$e^{-n} \sum_{k=0}^{2n} \frac{n^k}{k!} \geqslant 1 - \varepsilon$$

for all $n \geqslant n_0$. Since the mapping $t \mapsto \mu_t$ from $\mathbb{R}_+$ into $M^1(E)$ is continuous, the set $\{\mu_t : t \in [0, 2]\}$ is relatively compact. But then Prohorov's theorem 1.3.7 applies, and there exists a set $K \in \mathcal{K}(E)$ with

$$\mu_t(K) \geqslant 1 - \varepsilon$$

for all $t \in [0, 2]$. This implies that for all $n \geqslant n_0$

$$e(n\mu_{\frac{1}{n}})(K) = e^{-n} \sum_{k \geqslant 0} \frac{n^k}{k!} \mu_{\frac{k}{n}}(K) \geqslant e^{-n} \sum_{k=0}^{2n} \frac{n^k}{k!} \mu_{\frac{k}{n}}(K)$$

$$\geqslant e^{-n} \left(\sum_{k=0}^{2n} \frac{n^k}{k!}\right)(1-\varepsilon) \geqslant (1-\varepsilon)^2.$$

3. For all $z \in \mathbb{C}$ one has

$$n\left(\mathrm{e}^{\frac{z}{n}} - 1\right) \to z$$

as $n \to \infty$. It follows that

$$\lim_{n\to\infty} \mathrm{e}(n\mu_{\frac{1}{n}})^{\wedge}(a) = \lim_{n\to\infty} \exp((n\mu_{\frac{1}{n}})^{\wedge}(a) - n)$$

$$= \lim_{n\to\infty} \exp\left(n\left[\exp\left(\frac{1}{n}\operatorname{Log}\hat{\mu}(a)\right) - 1\right]\right)$$

$$= \exp(\operatorname{Log}\hat{\mu}(a)) = \hat{\mu}(a)$$

whenever $a \in E'$. We now apply 2. in order to use the continuity theorem 2.1.9 yielding

$$\lim_{n\to\infty} \mathrm{e}(n\mu_{\frac{1}{n}}) = \mu\,.$$

<div align="right">□</div>

Theorem 2.4.12 (Continuity of the Poisson mapping)
Let $(\lambda_n)_{n\geqslant 1}$ be a sequence of measures in $M^{\mathrm{b}}(E)$.

(i) *If $(\lambda_n)_{n\geqslant 1}$ is τ_{w}-relatively compact, then so is $(\mathrm{e}(\lambda_n))$.*
(ii) *The mapping $\mathrm{e} : M^{\mathrm{b}}(E) \to M^1(E)$ is continuous in the sense that the τ_{w}-convergence $\lambda_n \to \lambda \in M^{\mathrm{b}}(E)$ implies the τ_{w}-convergence*

$$\mathrm{e}(\lambda_n) \to \mathrm{e}(\lambda)$$

as $n \to \infty$.

Proof. (i). By Prohorov's theorem 1.3.7

$$d := \sup_{n\geqslant 1} \|\lambda_n\| < \infty\,.$$

For given $\varepsilon > 0$ there exists a $k_0 \in \mathbb{N}$ satisfying

$$\sum_{k > k_0} \frac{d^k}{k!} \leqslant \frac{\varepsilon}{2}\,.$$

But since $\{\lambda_n^k : 0 \leqslant k \leqslant k_0, n \in \mathbb{N}\}$ is τ_{w}-relatively compact (by the continuity of the convolution) Prohorov's theorem 1.3.7 provides us with the existence of a set $K \in \mathcal{K}(E)$ such that

$$\lambda_n^k(\complement K) \leqslant \frac{\varepsilon}{2\mathrm{e}}$$

whenever $0 \leqslant k \leqslant k_0$, $n \in \mathbb{N}$. Therefore

$$e(\lambda_n)(\mathbb{C}K) = e^{-\|\lambda_n\|} \sum_{k \geqslant 0} \frac{1}{k!} \lambda_n^k(\mathbb{C}K)$$

$$\leqslant \sum_{k=0}^{k_0} \frac{1}{k!} \lambda_n^k(\mathbb{C}K) + \sum_{k > k_0} \frac{1}{k!} \lambda_n^k(E)$$

$$\leqslant \frac{\varepsilon}{2e} \sum_{k=0}^{k_0} \frac{1}{k!} + \sum_{k > k_0} \frac{d^k}{k!} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Another application of Prohorov's theorem 1.3.7 yields (i).

(ii). We apply Property 2.4.10.1 in order to obtain

$$\lim_{n \to \infty} e(\lambda_n)^\wedge(a) = e(\lambda)^\wedge(a)$$

whenever $a \in E'$. But then (i) together with the continuity theorem 2.1.9 implies the assertion. $\square$

Remark 2.4.13 The converse of statement (i) of Theorem 2.4.12 does not hold in general. In fact, by Theorem 2.4.11 we have for any $\mu \in I(E)$ that

$$e(n\mu_{\frac{1}{n}}) \to \mu$$

as $n \to \infty$, but

$$\sup_{n \geqslant 1} \|n\mu_{\frac{1}{n}}\| = \sup_{n \geqslant 1} (n\mu_{\frac{1}{n}})(E)$$

$$= \sup_{n \geqslant 1} n = \infty.$$

Résumé 2.4.14. *A measure $\mu \in M^1(E)$ is called (continuously) **embeddable** if there exists a (continuous) convolution semigroup $(\mu_t)_{t \in \mathbb{R}_+}$ in $M^1(E)$ such that $\mu_1 = \mu$.*

By $EM(E)$ we denote the totality of all embeddable measures on E. Then $EM(E)$ is a subsemigroup of $M^1(E)$, and $G(E)$ as well as $P(E)$ are subsemigroups of the semigroup $I(E)$ which by the embedding Theorem 2.3.9 is contained in $EM(E)$. While $I(E)$ is τ_w-closed in $M^1(E)$ by Theorem 2.3.7, $P(E)$ is τ_w-dense in $I(E)$ by Theorem 2.4.11.

Chapter 3

The Structure of Infinitely Divisible Probability Measures

3.1 The Ito–Nisio theorem

This section is devoted to studying the convergence behaviour of sums of independent random variables taking their values in a separable Banach space E. The discussion will provide a complete version of Lévy's continuity theorem which extends Theorem 2.1.9.

Definition 3.1.1 A subset Z of E is said to be a (measurable) **cylinder set** if there exist a finite set $\{a_1, \ldots, a_n\}$ in E' and a set $B \in \mathfrak{B}(\mathbb{R}^n)$ such that

$$Z = Z(a_1, \ldots, a_n; B) := \{x \in E : (\langle x, a_1 \rangle, \ldots, \langle x, a_n \rangle) \in B\}.$$

Let $\mathfrak{Z}(E)$ denote the system of cylinder sets of E.

Properties 3.1.2 of cylinder sets.

3.1.2.1 $\mathfrak{Z}(E) \subset \mathfrak{B}(E)$.

3.1.2.2 $\mathfrak{Z}(E) = \mathfrak{B}(E)$ *if and only if* $\dim E < \infty$.

3.1.2.3 $\mathfrak{Z}(E)$ *is an algebra.*

In fact, we note that $E = Z(a; \mathbb{R})$ for arbitrary $a \in E'$ and hence belongs to $\mathfrak{Z}(E)$. Moreover, for $Z := Z(a_1, \ldots, a_n; B) \in \mathfrak{Z}(E)$ also $\complement Z = Z(a_1, \ldots, a_n; \complement B) \in \mathfrak{Z}(E)$ and with $Z' := Z(a'_1, \ldots, a'_m; B')$ for $\{a'_1, \ldots, a'_m\}, B' \in \mathfrak{B}(\mathbb{R})^m$ we have that

$$Z \cap Z' = Z(a_1, \ldots, a_n, a'_1, \ldots, a'_m; B \times B') \in \mathfrak{Z}(E).$$

3.1.2.4 $\mathfrak{Z}(E)$ *is translation invariant.*

This property follows directly from

$$Z + y = Z(a_1, \ldots, a_n; B + (\langle y, a_1 \rangle, \ldots, \langle y, a_n \rangle))$$

for $Z := Z(a_1, \ldots, a_n; B)$ with $\{a_1, \ldots, a_n\} \subset E'$, $B \in \mathfrak{B}(\mathbb{R})^n$ and $y \in E$.

3.1.2.5 $\sigma(\mathfrak{Z}(E)) = \mathfrak{B}(E)$

By Property 3.1.2.1 it suffices to show that $\mathcal{O}(E) \subset \sigma(\mathfrak{Z}(E))$. Moreover, applying the assumption that E is separable the proof is reduced to showing that $B(x, r)^- \in \sigma(\mathfrak{Z}(E))$ for all $x \in E$, $r \in \mathbb{R}_+^\times$. Finally, by Property 3.1.2.4 it remains to verify that $B(0, r)^- \in \sigma(\mathfrak{Z}(E))$ for every $r \in \mathbb{R}_+^\times$.

Let $(x_n)_{n \geqslant 1}$ be a sequence dense in E. From Appendix B.5 (Banach, Hahn) we obtain a sequence $(a_n)_{n \geqslant 1}$ in E' with $\|a_n\| = 1$ such that $\langle x_n, a_n \rangle = \|x_n\|$ for all $n \geqslant 1$. Now, let

$$B_r := \bigcap_{n \geqslant 1} \{x \in E : \langle x, a_n \rangle \leqslant r\} .$$

Clearly, $B_r \in \sigma(\mathfrak{Z}(E))$ and hence the proof is finished once $B(0, r)^- = B_r$ has been established. At first we note that $B(0, r) \subset B_r$. For the reversed inclusion we pick $x \in B_r$ to which there exists a subsequence $(x_{n_k})_{k \geqslant 1}$ of $(x_n)_{n \geqslant 1}$ such that $\lim_{k \to \infty} x_{n_k} = x$. But the inequalities

$$\|x_{n_k}\| = \langle x_{n_k}, a_{n_k} \rangle = \langle x_{n_k} - x, a_{n_k} \rangle + \langle x, a_{n_k} \rangle \leqslant \|x_{n_k} - x\| + r$$

valid for all $k \geqslant 1$ imply that $\|x\| \leqslant r$, hence that $x \in B(0, r)^-$.

Discussion 3.1.3 of types of convergence for E-valued random variables. Let $(Y_n)_{n \geqslant 1}$ be a sequence of E-valued random variables on a probability space $(\Omega, \mathfrak{A}, \mathsf{P})$.

3.1.3.1 $(Y_n)_{n \geqslant 1}$ converges P-a.s. (in symbols $Y_n \xrightarrow{\mathsf{P}\text{-a.s.}}$) if and only if for all $\varepsilon > 0$

$$\lim_{m \to \infty} \mathsf{P}\left(\left[\sup_{n \geqslant m} \|Y_n - Y_m\| > \varepsilon\right]\right) = 0 .$$

3.1.3.2 $(Y_n)_{n \geqslant 1}$ converges P-stochastically (in symbols $Y_n \xrightarrow{\mathsf{P}\text{-stoch}}$) if and only if for all $\varepsilon > 0$

$$\lim_{m \to \infty} \sup_{n \geqslant m} \mathsf{P}([\|Y_n - Y_m\| > \varepsilon]) = 0 .$$

3.1.3.3 If Y denotes the P-a.s. or P-stoch limit of $(Y_n)_{n \geqslant 1}$ then the limiting relations hold for Y instead of Y_m in 3.1.3.1 and 3.1.3.2 respectively.

We note that P-a.s. convergence implies P-stoch convergence, and P-stoch convergence implies convergence of $(Y_n)_{n \geqslant 1}$ in distribution which is defined as weak convergence of the sequence $(\mathsf{P}_{Y_n})_{n \geqslant 1}$ of distributions P_{Y_n} of Y_n as measures in $M^1(E)$ (in symbols $Y_n \overset{\mathrm{d}}{\longrightarrow}$).

Theorem 3.1.4 (Ottaviani's inequality)
Let $(Y_k)_{1 \leqslant k \leqslant n}$ be a sequence of independent E-valued random variables on $(\Omega, \mathfrak{A}, \mathsf{P})$. Then for all $\varepsilon > 0$ we have the inequality

$$\left(1 - \max_{1 \leqslant k < n} \mathsf{P}([\|Y_{k+1} + \cdots + Y_n\| \geqslant \varepsilon]) \right) \mathsf{P}\left(\left[\max_{1 \leqslant k < n} \|Y_1 + \cdots + Y_k\| > 2\varepsilon \right] \right)$$

$$\leqslant \mathsf{P}([\|Y_1 + \cdots + Y_n\| > \varepsilon]).$$

Proof.   For every $k = 1, \ldots, n-1$ let

$$B_k := [\|Y_1\| < 2\varepsilon, \ldots, \|Y_1 + \cdots + Y_{k-1}\| \leqslant 2\varepsilon, \|Y_1 + \cdots + Y_k\| > 2\varepsilon],$$

$$C_k := [\|Y_{k+1} + \cdots + Y_n\| < \varepsilon], \quad \text{and}$$

$$D := [\|Y_1 + \cdots + Y_n\| > \varepsilon].$$

Then $B_1, \ldots, B_{n-1}$ are pairwise disjoint, and B_k, C_k are independent for $1 \leqslant k \leqslant n-1$. Since

$$\|Y_1 + \cdots + Y_n\| \geqslant \|Y_1 + \cdots + Y_k\| - \|Y_{k+1} + \cdots + Y_n\|,$$

we obtain

$$\bigcup_{k=1}^{n-1} (B_k \cap C_k) \subset D,$$

and hence

$$\mathsf{P}(D) \geqslant \sum_{k=1}^{n-1} \mathsf{P}(B_k \cap C_k) = \sum_{k=1}^{n} \mathsf{P}(B_k)\,\mathsf{P}(C_k)$$

$$\geqslant \left(\min_{1 \leqslant k < n} \mathsf{P}(C_k) \right) \left(\sum_{k=1}^{n-1} \mathsf{P}(B_k) \right)$$

$$= \min_{1 \leqslant k < n} \left(1 - \mathsf{P}\left(\complement C_k \right) \right) \mathsf{P}\left(\bigcup_{k=1}^{n-1} B_k \right)$$

$$= \left(1 - \max_{1 \leqslant k < n} \mathsf{P}\left(\complement C_k \right) \right) \cdot \mathsf{P}([\max_{1 \leqslant k < n} \|Y_1 + \cdots + Y_k\| > 2\varepsilon]). \qquad \square$$

Corollary 3.1.5 *For each $\ell \geqslant 1$ let $S_\ell := \sum_{j=1}^{\ell} Y_j$. Then*

$$\left(1 - \sup_{n \geqslant 1} \max_{1 \leqslant k < n} \mathsf{P}([\|S_n - S_k\| \geqslant \varepsilon]) \right) \mathsf{P}\left(\left[\sup_{k \geqslant 1} \|S_k\| > 2\varepsilon \right] \right)$$

$$\leqslant \sup_{n \geqslant 1} \mathsf{P}([\|S_n\| > \varepsilon])$$

for all $\varepsilon > 0$.

Proof.   Applying the theorem to each of the sequences $(Y_k)_{1 \leqslant k \leqslant n}$ $(n \geqslant 1)$ we obtain for all $\varepsilon > 0$

$$\left(1 - \sup_{n \geqslant 1} \max_{1 \leqslant k < n} \mathsf{P}([\|S_n - S_k\| \geqslant \varepsilon])\right) \mathsf{P}\left(\left[\max_{1 \leqslant k < m} \|S_k\| > 2\varepsilon\right]\right)$$

$$\leqslant \sup_{n \geqslant 1} \mathsf{P}([\|S_n\| > \varepsilon])$$

whenever $m \geqslant 1$.   But we have

$$\left[\max_{1 \leqslant k < m} \|S_k\| > 2\varepsilon\right] \uparrow \left[\sup_{k \geqslant 1} \|S_k\| > 2\varepsilon\right],$$

hence by the continuity from below of the measure P the assertion follows.

□

Theorem 3.1.6   (Equivalence of types of convergence)
Let $(X_k)_{k \geqslant 1}$ be a sequence of independent E-valued random variables on $(\Omega, \mathfrak{A}, \mathsf{P})$ with distributions $\lambda_k = \mathsf{P}_{X_k}$ $(k \geqslant 1)$, and let $(S_n)_{n \geqslant 1}$ be the corresponding sequence of n-th partial sums $S_n = \sum_{k=1}^n X_k$ $(n \geqslant 1)$. For every $n \geqslant 1$ we abbreviate

$$\mu_n := \mathsf{P}_{S_n} = \lambda_1 * \cdots * \lambda_n.$$

Then the following statements are equivalent:

(i)   *$(S_n)_{n \geqslant 1}$ converges P-a.s.*
(ii)   *$(S_n)_{n \geqslant 1}$ converges P-stoch.*
(iii)   *$(S_n)_{n \geqslant 1}$ converges in distribution.*

Proof.   (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are well-known implications valid for arbitrary sequences of E-valued random variables on $(\Omega, \mathfrak{A}, \mathsf{P})$.   (For the second implication see Application 1.2.15.)

(ii) $\Rightarrow$ (i).   Let $\varepsilon > 0$.   For $\delta \in]0, \frac{1}{2}]$ there exists an $m_0 \geqslant 1$ such that

$$\mathsf{P}([\|X_{m+1} + \cdots + X_n\| \geqslant \varepsilon]) = \mathsf{P}([\|S_n - S_m\| \geqslant \varepsilon]) < \delta$$

whenever $m_0 \leqslant m < n$.   But Corollary 3.1.5 applied to the sequence of E-valued random variables $Y_j := X_{m+j}$ $(j \geqslant 1)$ yields

$$\frac{1}{2} \mathsf{P}\left(\left[\sup_{k \geqslant 1} \|S_{m+k} - S_m\| > 2\varepsilon\right]\right)$$

$$\leqslant (1 - \delta) \mathsf{P}\left(\left[\sup_{k \geqslant 1} \|S_{m+k} - S_m\| > 2\varepsilon\right]\right)$$

$$\leqslant \left(1 - \sup_{n \geqslant 1} \max_{1 \leqslant k < n} \mathsf{P}([\|S_{m+n} - S_{m+k}\| > \varepsilon])\right) \mathsf{P}\left(\left[\sup_{k \geqslant 1} \|S_{m+k} - S_m\| > 2\varepsilon\right]\right)$$

$$\leqslant \sup_{n \geqslant 1} \mathsf{P}([\|S_{m+n} - S_m\| > \varepsilon]) \leqslant \delta$$

(by Corollary 3.1.5), hence

$$\frac{1}{2} \, \mathsf{P} \left(\left[\sup_{n \geqslant m} \| S_n - S_m \| > 2\varepsilon \right] \right) \leqslant \delta$$

for all $m \geqslant m_0$, and this implies (i).

(iii) $\Rightarrow$ (ii). For $1 \leqslant m < n$ we have

$$\mu_{m,n} := \lambda_{m+1} * \cdots * \lambda_n = \mathsf{P}_{S_n - S_m} = \mathsf{P}_{X_{m+1} + \cdots + X_n}$$

and consequently

$$\mathsf{P}([\|S_n - S_m\| \geqslant \varepsilon]) = \mu_{m,n} \big(\complement B(0, \varepsilon) \big) \, .$$

Now we assume that $S_n \xrightarrow{\text{P-stoch}}$ is not fulfilled. Then there exist subsequences $(m_k)_{k \geqslant 1}$, $(n_k)_{k \geqslant 1}$ in $\mathbb{N}$ with $m_k < n_k$ and $\mu_{m_k, n_k} \big(\complement B(0, \varepsilon) \big) \geqslant \varepsilon$ for all $k \geqslant 1$. But since $\mu_{m_k} * \mu_{m_k, n_k} = \mu_{n_k}$ for all $k \geqslant 1$ and $\tau_{\mathrm{w}}-\lim_{n \to \infty} \mu_n = \mu \in M^1(E)$ by assumption, the sequence $(\mu_{m_k, n_k})_{k \geqslant 1}$ is τ_{w}-relatively compact. Without loss of generality we may therefore assume that $\tau_{\mathrm{w}}-\lim_{k \to \infty} \mu_{m_k, n_k} = \nu \in M^1(E)$. But then $\mu * \nu = \mu$ by (iii) of Theorem 1.4.9 and hence $\nu = \varepsilon_0$ by Lemma 2.2.2. Finally, the sequence of inequalities

$$0 = \varepsilon_0 \big(\complement B(0, \varepsilon) \big) \geqslant \limsup_{k \to \infty} \mu_{m_k, n_k} \big(\complement B(0, \varepsilon) \big) \geqslant \varepsilon > 0$$

yields the desired contradiction, and (ii) of the theorem has been established. $\qquad \square$

Corollary 3.1.7 *Let the sequence $(\mu_n)_{n \geqslant 1}$ of distributions P_{S_n} be shift tight in $M^1(E)$. Then there exists a sequence $(x_n)_{n \geqslant 1}$ in E such that*

$$S_n - x_n \mathbb{1}_\Omega \xrightarrow{\text{P-a.s.}} \, .$$

Proof. By Theorem 2.2.25 the sequence $(\nu_n)_{n \geqslant 1}$ with $\nu_n := |\mu_n|^2$ for all $n \geqslant 1$ is τ_{w}-relatively compact. Since $|\hat{\nu}_n| = |\hat{\mu}_n|^2$ by Property 2.2.24.3 and $\nu_n \prec \nu_{n+1}$, Property 2.2.16.7 yields $0 \leqslant \hat{\nu}_{n+1} \leqslant \hat{\nu}_n$ for all $n \geqslant 1$. By the continuity theorem 2.1.9 this implies that $(\nu_n)_{n \geqslant 1}$ is τ_{w}-convergent.

In the following we aim at representing the sequence $(\nu_n)_{n \geqslant 1}$ as a sequence of distributions of E-valued random variables on an appropriate probability space. Let

$$Y_n(\omega_1, \omega_2) := X_n(\omega_1)$$

and

$$Z_n(\omega_1, \omega_2) := X_n(\omega_2)$$

whenever $(\omega_1, \omega_2) \in \Omega \times \Omega$ $(n \geqslant 1)$. Clearly, $\{Y_1, Z_1, Y_2, Z_2, \dots\}$ is a sequence of independent E-valued random variables on the probability space $(\Omega \times \Omega, \mathfrak{A} \otimes \mathfrak{A}, \mathsf{P} \otimes \mathsf{P})$ such that

$$(\mathsf{P} \otimes \mathsf{P})_{Y_n} = (\mathsf{P} \otimes \mathsf{P})_{Z_n} = \mathsf{P}_{X_n} = \lambda_n$$

for all $n \geqslant 1$. As a consequence we obtain that the E-valued random variable

$$\sum_{k=1}^{n} (Y_k - Z_k) = \left(\sum_{k=1}^{n} Y_k \right) - \left(\sum_{k=1}^{n} Z_k \right)$$

has ν_n as its distribution $(n \geqslant 1)$ (see Application 1.4.5 and Property 1.4.4.4). It follows that

$$\sum_{k=1}^{n} (Y_k - Z_k) \xrightarrow{\;\mathrm{d}\;} \qquad \text{as } n \to \infty.$$

Moreover, since $(Y_n - Z_n)_{n \geqslant 1}$ is a sequence of independent E-valued random variables, the theorem applies and

$$\sum_{k=1}^{n} (Y_k - Z_k) \xrightarrow{\;\mathsf{P} \otimes \mathsf{P}\text{-a.s.}\;} W$$

where W denotes an E-valued random variable on $(\Omega \times \Omega, \mathfrak{A} \otimes \mathfrak{A}, \mathsf{P} \otimes \mathsf{P})$. But then there exists $Q \in \mathfrak{A} \otimes \mathfrak{A}$ with $(\mathsf{P} \otimes \mathsf{P})(Q) = 1$ such that

$$\lim_{n \to \infty} (S_n(\omega_1) - S_n(\omega_2)) = W(\omega_1, \omega_2)$$

for all $(\omega_1, \omega_2) \in Q$. An application of Fubini's theorem yields

$$1 = (\mathsf{P} \otimes \mathsf{P})(Q) = \int \mathsf{P}(Q_{\omega_2}) \, \mathsf{P}(\mathrm{d}\omega_2)$$

and hence provides us with an $\omega_2 \in \Omega$ such that $\mathsf{P}(Q_{\omega_2}) = 1$. Choosing $x_n := S_n(\omega_2)$ for all $n \geqslant 1$ we obtain the sequence $(x_n)_{n \geqslant 1}$ in E required in the assertion. $\qquad \square$

The following two Lemmata are designed to prepare the main result of the section.

Lemma 3.1.8 *Let X, Y, Y_1, Y_2, … be E-valued random variables on $(\Omega, \mathfrak{A}, \mathsf{P})$. For every $n \geqslant 1$ the random variables X and Y_n are assumed to be independent. Moreover, by hypothesis*

$$\langle Y_n, a \rangle \xrightarrow{\;\mathsf{P}\text{-stoch}\;} \langle Y, a \rangle$$

for every $a \in E'$. Then X and Y are independent.

Proof. Let $a_1, \ldots, a_k \in E'$. For each $n \geqslant 1$ we introduce the $\mathbb{R}^k$-valued random variable

$$\varphi_n := (\langle Y_n, a_1 \rangle, \ldots, \langle Y_n, a_k \rangle).$$

Then $\varphi_n \xrightarrow{\text{P-stoch}} \varphi := (\langle Y, a_1 \rangle, \ldots, \langle Y, a_k \rangle)$, since

$$\mathsf{P}([\|\varphi_n - \varphi\|_\infty > \varepsilon]) \leqslant \sum_{j=1}^{k} \mathsf{P}([|\langle Y_n, a_j \rangle - \langle Y, a_j \rangle| > \varepsilon]),$$

where by hypothesis the right-hand side tends to 0 for all $\varepsilon > 0$.

Next we note that by assumption the random variables

$$\psi := (\langle X, a_1 \rangle, \ldots, \langle X, a_k \rangle)$$

and φ_n are independent, and

$$\varphi_n \otimes \psi \xrightarrow{\text{P-stoch}} \varphi \otimes \psi.$$

The equivalence theorem 3.1.6 together with Theorem 1.4.8 yields

$$\mathsf{P}_{\varphi \otimes \psi} = \tau_{\mathrm{w}}\text{-}\lim_{n \to \infty} \mathsf{P}_{\varphi_n \otimes \psi} = \tau_{\mathrm{w}}\text{-}\lim_{n \to \infty} (\mathsf{P}_{\varphi_n} \otimes \mathsf{P}_\psi) = \mathsf{P}_\varphi \otimes \mathsf{P}_\psi,$$

thus φ and ψ are independent.

Now, let $Z_1, Z_2 \in \mathfrak{Z}(E)$. There exist $a_1, \ldots, a_k \in E'$ and $B_1, B_2 \in \mathfrak{B}(\mathbb{R}^k)$ such that

$$Z_j = Z_j(a_1, \ldots, a_k; B_j)$$

for $j = 1, 2$. From the above chain of equalities we deduce

$$\begin{aligned}
\mathsf{P}([Y \in Z_1, X \in Z_2]) &= \mathsf{P}([\varphi \in B_1, \psi \in B_2]) \\
&= \mathsf{P}([\varphi \in B_1]) \, \mathsf{P}([\psi \in B_2]) \\
&= \mathsf{P}([Y \in Z_1]) \, \mathsf{P}([X \in Z_2]).
\end{aligned}$$

Since $\mathfrak{Z}(E)$ is a $\cap$-stable generator of $\mathfrak{B}(E)$ by Properties 3.1.2.3 and 3.1.2.5 we have shown that X and Y are independent. $\qquad\square$

Lemma 3.1.9 *Let* $(\Omega, \mathfrak{A}, \mathsf{P}, (Y_a)_{a \in E'})$ *and* $(E, \mathfrak{B}(E), \mu, (\langle \cdot, a \rangle)_{a \in E'})$ *denote two equivalent stochastic processes (with parameter set E' and state space $\mathbb{R}$). Then there exists an E-valued random variable Y on $(\Omega, \mathfrak{A}, \mathsf{P})$ such that*

$$\mathsf{P}([Y_a = \langle Y, a \rangle]) = 1$$

whenever $a \in E'$.

In particular, $\mathsf{P}_Y = \mu$.

Proof.   With $\mathbb{R}^{\mathbb{N}}$ carrying the product topology we have that $\mathfrak{B}(\mathbb{R}^{\mathbb{N}}) = \mathfrak{B}(\mathbb{R})^{\otimes \mathbb{N}}$. Let $(a_n)_{n \geqslant 1}$ be a sequence in E' with

$$\bigcap_{n \geqslant 1} \{x \in E : \langle x, a_n \rangle \leqslant r\} = B(0, r)^-$$

for every $r \in \mathbb{R}_+^{\times}$.   This follows from Appendix B.5 (Banach, Hahn) which indeed yields a sequence $(a_n)_{n \geqslant 1}$ in E' satisfying $\|a_n\| = 1$ and $\langle x, a_n \rangle = \|x\|$ for all $n \geqslant 1$.

By

$$\varphi(x) := (\langle x, a_n \rangle)_{n \geqslant 1}$$

for all $x \in E$ we introduce an injective continuous linear mapping from E into $\mathbb{R}^{\mathbb{N}}$.   Now, $\mu \in M^1(E)$ is tight by Theorem 1.1.6.   Hence there exists an increasing sequence $(K_n)_{n \geqslant 1}$ in $\mathcal{K}(E)$ such that for $C := \bigcup_{n \geqslant 1} K_n$ we obtain that $\mu(C) = 1$.   Clearly, $C \in \mathfrak{B}(E)$ and $\varphi(C) = \bigcup_{n \geqslant 1} \varphi(K_n) \in \mathfrak{B}(\mathbb{R}^{\mathbb{N}})$.   But then $\mathrm{Res}_{K_n} \varphi$ is a homeomorphism from K_n onto $\varphi(K_n)$ and hence $\mathrm{Res}_C \varphi$ turns out to be a bimeasurable bijection from $(C, C \cap \mathfrak{B}(E))$ onto $(\varphi(C), \varphi(C) \cap \mathfrak{B}(\mathbb{R}^{\mathbb{N}}))$.   Consequently, there is a measurable mapping ψ from $(\mathbb{R}^{\mathbb{N}}, \mathfrak{B}(\mathbb{R}^{\mathbb{N}}))$ into $(E, \mathfrak{B}(E))$ given by $\psi(\varphi(x)) = x$ for all $x \in C$.   Now we put

$$\Lambda(\omega) := (Y_{a_n}(\omega))_{n \geqslant 1}$$

for all $\omega \in \Omega$. Λ is a measurable map from $(\Omega, \mathfrak{A})$ into $(\mathbb{R}^{\mathbb{N}}, \mathfrak{B}(\mathbb{R}^{\mathbb{N}}))$. For every finite sequence $\{B_1, \ldots, B_k\}$ in $\mathfrak{B}(\mathbb{R})$ we have by assumption that

$$\begin{aligned}
\mathsf{P}([\Lambda &\in B_1 \times \cdots \times B_k \times \mathbb{R} \times \mathbb{R} \times \cdots]) \\
&= \mathsf{P}([Y_{a_j} \in B_j \text{ for } j = 1, \ldots, k]) \\
&= \mu([\langle \cdot, a_j \rangle \in B_j \text{ for } j = 1, \ldots, k]) \\
&= \mu([\varphi \in B_1 \times \cdots \times B_k \times \mathbb{R} \times \mathbb{R} \times \cdots]).
\end{aligned}$$

But the system of sets of the form $B_1 \times \cdots \times B_k \times \mathbb{R} \times \mathbb{R} \times \cdots$ is a $\cap$-stable generator of $\mathfrak{B}(\mathbb{R}^{\mathbb{N}}) = (\mathfrak{B}(\mathbb{R}))^{\otimes \mathbb{N}}$, hence $\mathsf{P}_\Lambda = \mu_\varphi$.

Analogously one shows that

$$\mathsf{P}_{Y_a \otimes \Lambda} = \mu_{\langle \cdot, a \rangle \otimes \varphi}$$

for all $a \in E'$.

Now, for $a \in E'$ we form the set

$$B_a := \{(t_n)_{n \geqslant 0} \in \mathbb{R} \times \mathbb{R}^{\mathbb{N}} : t_0 = \langle \psi((t_n)_{n \geqslant 1}), a \rangle\} \in \mathfrak{B}(\mathbb{R} \times \mathbb{R}^{\mathbb{N}})$$

and introduce the E-valued random variable $Y := \psi \circ \Lambda$. Then

$$
\begin{aligned}
\mathsf{P}([Y_a = \langle Y, a \rangle]) &= \mathsf{P}([Y_a = \langle \psi \circ \Lambda, a \rangle]) \\
&= \mathsf{P}([Y_a \otimes \Lambda \in B_a]) \\
&= \mu([\langle \cdot, a \rangle \otimes \varphi \in B_a]) \\
&= \mu([\langle \cdot, a \rangle = \langle \psi \circ \varphi, a \rangle]) \\
&= \mu([\langle \cdot, a \rangle = \langle \psi \circ \varphi, a \rangle] \cap C) \\
&= \mu(C) = 1 \,,
\end{aligned}
$$

and the first statement of the Lemma has been established.

For the remaining statement we take

$$
Z := Z(a_1, \ldots, a_n; B) \in \mathfrak{Z}(E)
$$

(for $a_1, \ldots, a_n \in E'$, $B \in \mathfrak{B}(\mathbb{R}^n)$). Then

$$
\begin{aligned}
\mathsf{P}_Y(Z) &= \mathsf{P}([(\langle Y, a_1 \rangle, \ldots, \langle Y, a_n \rangle) \in B]) \\
&= \mathsf{P}([Y_{a_1} \otimes \cdots \otimes Y_{a_n} \in B]) \\
&= \mu(Z) \,.
\end{aligned}
$$

Applying Properties 3.1.2.3 and 3.1.2.5 this implies that $\mathsf{P}_Y = \mu$. $\qquad\square$

Theorem 3.1.10 (Ito, Nisio)
Let $(X_k)_{k \geqslant 1}$ be a sequence of independent E-valued random variables X_k on $(\Omega, \mathfrak{A}, \mathsf{P})$ with the property that P_{X_k} is symmetric for all $k \geqslant 1$, and let $(S_n)_{n \geqslant 1}$ be the corresponding sequence of n-th partial sums $\sum_{k=1}^n X_k$.
The following statements are equivalent:

(i) *$(S_n)_{n \geqslant 1}$ converges P-almost surely.*
(ii) *$(S_n)_{n \geqslant 1}$ converges P-stochastically.*
(iii) *$(S_n)_{n \geqslant 1}$ converges in distribution.*
(iv) *$(\mathsf{P}_{S_n})_{n \geqslant 1}$ is uniformly tight.*
(v) *There exists an E-valued random variable S on $(\Omega, \mathfrak{A}, \mathsf{P})$ such that*

$$
\langle S_n, a \rangle \xrightarrow{\ \mathsf{P}\text{-stoch}\ } \langle S, a \rangle
$$

for all $a \in E'$.
(vi) *There exists a measure $\mu \in M^1(E)$ such that*

$$
\hat{\mathsf{P}}_{S_n}(a) \to \hat{\mu}(a)
$$

for all $a \in E'$.

Proof.   From the equivalence theorem 3.1.6 we deduce the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) and from the continuity theorem 2.1.9 the implication (iii) $\Rightarrow$ (iv).

We proceed to the proof of implication (iv) $\Rightarrow$ (i). By assumption the sequence $(P_{S_n})_{n \geqslant 1}$ is uniformly tight, consequently Corollary 3.1.7 becomes applicable and hence there exists a sequence $(x_n)_{n \geqslant 1}$ in E such that

$$S_n - x_n \mathbb{1}_\Omega \xrightarrow{\text{P-a.s.}}$$

as $n \to \infty$. Since the random variables X_k are independent and their distributions P_{X_n} are symmetric for all $k \geqslant 1$, we obtain

$$P_{\bigotimes_{n \geqslant 1} X_n} = \bigotimes_{n \geqslant 1} P_{X_n} = \bigotimes_{n \geqslant 1} P_{X_n}^- = \bigotimes_{n \geqslant 1} P_{-X_n} = P_{\bigotimes_{n \geqslant 1} (-X_n)},$$

and therefore for every $\varepsilon > 0$ and all $k \geqslant 1$

$$P\left(\left[\sup_{m \geqslant k} \|(S_m - x_m) - (S_k - x_k)\| > \varepsilon\right]\right)$$

$$= P\left(\left[\sup_{m \geqslant k} \|X_{k+1} + \cdots + X_m - (x_m - x_k)\| > \varepsilon\right]\right)$$

$$= P_{\bigotimes_{n \geqslant 1} X_n}\left(\left\{(y_n)_{n \geqslant 1} \in E^{\mathbb{N}} : \sup_{m \geqslant k} \|y_{k+1} + \cdots + y_m - (x_m - x_k)\| > \varepsilon\right\}\right)$$

$$= P_{\bigotimes_{n \geqslant 1} (-X_n)}\left(\left\{(y_n)_{n \geqslant 1} \in E^{\mathbb{N}} : \sup_{m \geqslant k} \|y_{k+1} + \cdots + y_m - (x_m - x_k)\| > \varepsilon\right\}\right)$$

$$= P\left(\left[\sup_{m \geqslant k} \| - X_{k+1} - \cdots - X_m - (x_m - x_k)\| > \varepsilon\right]\right)$$

$$= P\left(\left[\sup_{m \geqslant k} \|(-S_m - x_m) - (-S_k - x_k)\| > \varepsilon\right]\right).$$

But this implies the limiting relation

$$-S_n - x_n \mathbb{1}_\Omega \xrightarrow{\text{P-a.s.}}$$

as $n \to \infty$. Observing that for every $n \geqslant 1$

$$S_n = \frac{1}{2}(S_n - x_n \mathbb{1}_\Omega) - \frac{1}{2}(-S_n - x_n \mathbb{1}_\Omega)$$

we obtain the desired statement (i).

(v) $\Rightarrow$ (iv). Let $\mu := P_S$. By the inner regularity of μ (see Theorem 1.1.2 (ii)) for every $\varepsilon > 0$ there exists a $K \in \mathcal{K}(E)$ such that $\mu(\complement K) < \varepsilon$. The set $K_1 := \{\frac{1}{2}(x - y) : x, y \in K\}$ is compact. Now, given $k \geqslant 1$ we

note that for every $n \geqslant 1$ the E-valued random variables $X := S_k$ and $Y_n := S_{k+n} - S_k$ are independent, and by hypothesis

$$\langle Y_n, a \rangle = \langle S_{k+n}, a \rangle - \langle S_k, a \rangle \xrightarrow{\text{P-stoch}} \langle S, a \rangle - \langle S_k, a \rangle = \langle S - S_k, a \rangle$$

whenever $a \in E'$. Applying Lemma 3.1.8 this implies that the random variables X and $Y := S - S_k$ are independent. But with $X + Y = S$ and $\mu_k := \mathsf{P}_{S_k} = \mathop{\text{\Large$*$}}_{j=1}^{k} \mathsf{P}_{X_k}$ we obtain

$$1 - \varepsilon < \mu(K) = \mu_k * \mathsf{P}_{S - S_k}(K)$$
$$= \int \mu_k(K - x) \, \mathsf{P}_{S - S_k}(\mathrm{d}x) \,,$$

hence there exists an $x_k \in E$ such that $\mu_k(K - x_k) > 1 - \varepsilon$. This, however, implies that

$$\begin{aligned}
\mu_k\big(\complement K_1\big) &= \mathsf{P}([S_k \notin K_1]) \\
&\leqslant \mathsf{P}([S_k + x_k \notin K] \cup [-S_k + x_k \notin K]) \\
&\leqslant \mathsf{P}([S_k \notin K - x_k]) + \mathsf{P}([-S_k \notin K - x_k]) \\
&= 2\mu_k\big(\complement(K - x_k)\big) < 2\varepsilon \,.
\end{aligned}$$

Since the inequalities are valid for each $k \geqslant 1$, (iv) has been established.

(vi) $\Rightarrow$ (v). For each $a \in E'$ $(\langle X_n, a \rangle)_{n \geqslant 1}$ is a sequence of independent real-valued random variables on $(\Omega, \mathfrak{A}, \mathsf{P})$, and for all $n \geqslant 1$

$$\langle X_1, a \rangle + \cdots + \langle X_n, a \rangle = \langle S_n, a \rangle \,.$$

But then

$$\hat{\mathsf{P}}_{\langle S_n, a \rangle}(t) = a(\mathsf{P}_{S_n})^{\wedge}(t) = \hat{\mathsf{P}}_{S_n}(a^{\top}(t)) = \hat{\mathsf{P}}_{S_n}(ta)$$

and

$$\widehat{a(\mu)}(t) = \hat{\mu}(a^{\top}(t)) = \hat{\mu}(ta)$$

whenever $a \in E'$, $t \in \mathbb{R}$. Now, by hypothesis

$$\lim_{n \to \infty} \hat{\mathsf{P}}_{\langle S_n, a \rangle}(t) = \widehat{a(\mu)}(t)$$

for all $t \in \mathbb{R}$, and by the classical continuity theorem which follows from Theorem 4.3.8 we have

$$\mathsf{P}_{\langle S_n, a \rangle} \xrightarrow{\tau_{\mathrm{w}}} a(\mu)$$

or

$$\langle S_n, a \rangle \xrightarrow{\text{d}} \,.$$

The equivalence theorem 3.1.6 now yields

$$\langle S_n, a \rangle \xrightarrow{\text{P-a.s.}} Y_a,$$

where Y_a is a real-valued random variable on $(\Omega, \mathfrak{A}, \mathsf{P})$.

On the other hand the two stochastic processes

$$(\Omega, \mathfrak{A}, \mathsf{P}, (Y_a)_{a \in E'}) \quad \text{and} \quad (E, \mathfrak{B}(E), \mu, ((\langle \cdot, a \rangle)_{a \in E'})$$

are equivalent.

In fact, let $a_1, \ldots, a_k \in E'$. Then for all $(t_1, \ldots, t_k) \in \mathbb{R}^k$ assumption (vi) together with an application of Lebesgue's dominated convergence theorem implies that

$$
\begin{aligned}
\hat{\mathsf{P}}_{(Y_{a_1}, \ldots, Y_{a_k})}(t_1, \ldots, t_k) &= \int \exp\left(\mathrm{i} \sum_{j=1}^{k} t_j Y_{a_j}\right) \mathrm{d}\mathsf{P} \\
&= \lim_{n \to \infty} \int \exp\left(\mathrm{i} \sum_{j=1}^{k} t_j \langle S_n, a_j \rangle\right) \mathrm{d}\mathsf{P} \\
&= \lim_{n \to \infty} \int \exp\left(\mathrm{i} \left\langle S_n, \sum_{j=1}^{k} t_j a_j \right\rangle\right) \mathrm{d}\mathsf{P} \\
&= \lim_{n \to \infty} \hat{\mathsf{P}}_{S_n}\left(\sum_{j=1}^{k} t_j a_j\right) = \int \exp\left(\mathrm{i} \sum_{j=1}^{k} t_j \langle \cdot, a_j \rangle\right) \mathrm{d}\mu \\
&= \hat{\mu}_{(\langle \cdot, a_1 \rangle, \ldots, \langle \cdot, a_k \rangle)}(t_1, \ldots, t_k)
\end{aligned}
$$

hence that

$$\mathsf{P}_{(Y_{a_1}, \ldots, Y_{a_k})} = \mu_{(\langle \cdot, a_1 \rangle, \ldots, \langle \cdot, a_k \rangle)}.$$

Finally, we apply Lemma 3.1.9 and obtain an E-valued random variable $S := Y$ on $(\Omega, \mathfrak{A}, \mathsf{P})$ such that

$$\mathsf{P}([Y_a = \langle S, a \rangle]) = 1$$

for all $a \in E'$. Consequently,

$$\langle S_n, a \rangle \xrightarrow{\text{P-a.s.}} \langle S, a \rangle,$$

thus

$$\langle S_n, a \rangle \xrightarrow{\text{P-stoch}} \langle S, a \rangle$$

whenever $a \in E'$. But this is (v). $\qquad \square$

Corollary 3.1.11 *Let* $(\mu_n)_{n\geqslant 1}$ *be a sequence of symmetric measures in* $M^1(E)$ *such that* $\mu_n \prec \mu_{n+1}$ *for all* $n \geqslant 1$, *and let* $\mu \in M^1(E)$ *be such that*

$$\lim_{n\to\infty} \hat\mu_n(a) = \hat\mu(a)$$

for all $a \in E'$. *Then*

$$\mu_n \xrightarrow{\tau_w} \mu\,.$$

Proof.   By assumption, for every $n \geqslant 2$ there exists a $\lambda_n \in M^1(E)$ with $\mu_n = \lambda_n * \mu_{n-1}$. Moreover we put $\lambda_1 := \mu_1$. Taking Fourier transforms and applying Properties 2.1.6.7 and 2.2.18.3 we see that λ_n is symmetric for all $n \geqslant 1$.

Now, let $(X_n)_{n\geqslant 1}$ be a sequence of independent E-valued random variables on $(\Omega, \mathfrak{A}, \mathsf{P})$ such that $\mathsf{P}_{X_n} = \lambda_n$ for all $n \geqslant 1$. Then the distribution of $S_n := \sum_{k=1}^n X_k$ has the form

$$\mathsf{P}_{S_n} = \underset{k=1}{\overset{n}{\bigast}}\,\mathsf{P}_{X_k} = \underset{k=1}{\overset{n}{\bigast}}\,\lambda_k = \mu_n\,.$$

Therefore (vi) of the theorem becomes applicable, and we obtain that

$$\mu_n \xrightarrow{\tau_w} \nu \in M^1(E)\,.$$

It remains to apply the uniqueness and continuity properties of the Fourier transform (Theorems 2.1.4 and 2.1.9 respectively) in order to see that $\nu = \mu$. □

Remark 3.1.12 Without the assumption of symmetry for the distributions P_{X_k} of the random variables X_k $(k \geqslant 1)$ the implication (v) $\Rightarrow$ (i) of the Ito–Nisio theorem 3.1.10 does not remain valid.

In fact, let E denote an infinite-dimensional separable Hilbert space with orthonormal basis $\{e_n : n \geqslant 1\}$. Let $(\Omega, \mathfrak{A}, \mathsf{P})$ be a probability space, and let

$$X_n(\omega) := e_n - e_{n-1}$$

for all $\omega \in \Omega$, $n \geqslant 1$ (where $e_0 := 0$). Then $(X_n)_{n\geqslant 1}$ is a sequence of independent E-valued random variables on $(\Omega, \mathfrak{A}, \mathsf{P})$. Moreover

$$S_n(\omega) = \sum_{k=1}^n X_k(\omega) = e_n$$

for all $\omega \in \Omega$, $n \geqslant 1$. Finally, we put $S(\omega) := 0$ for all $\omega \in \Omega$. Then

$$\lim_{n \to \infty} \langle S_n, a \rangle = \lim_{n \to \infty} \langle e_n, a \rangle = 0 = \langle S, a \rangle$$

whenever $a \in E'$, and this implies (v).

But

$$\lim_{n \to \infty} S_n(\omega) \neq S(\omega),$$

since $\|S_n(\omega)\| = 1$ for all $n \geqslant 1$ $(\omega \in \Omega)$, hence (i) does not hold.

As an immediate application of the Ito–Nisio theorem 3.1.10 we discuss the representation of an E-valued Gaussian random variable as an a.s.-convergent random series (in E).

Theorem 3.1.13 *Let ρ be a Gauss measure on E. Then there exist a sequence $(x_n)_{n \geqslant 1}$ in E and a sequence $(\xi_n)_{n \geqslant 1}$ of independent real-valued random variables on a probability space $(\Omega, \mathfrak{A}, \mathsf{P})$ with $\mathsf{P}_{\xi_n} = N(0,1)$ for all $n \in \mathbb{N}$ such that the series $\sum_{n \geqslant 1} \xi_n x_n$ is P-a.s. convergent, and*

$$\mathsf{P}_{\sum_{n \geqslant 1} \xi_n x_n} = \rho.$$

Proof. We recall the proof of Theorem 2.4.7 and realize that given $\rho \in \mathrm{G}(E)$ there is a continuous linear mapping $T : (E', \tau(E', E)) \to (L^2(E, \rho), \| \cdot \|)$ such that

$$\hat{\rho}(a) = \exp\left(-\frac{1}{2} \|Ta\|^2 \right)$$

for all $a \in E'$. Moreover, $T^\top$ can be regarded as a mapping from $L^2(E, \rho)$ into E. For a given orthonormal basis $\{f_n : n \in \mathbb{N}\}$ of $L^2(E, \rho)$ we define the sequence $(x_n)_{n \geqslant 1}$ in E by $x_n := T^\top f_n$ for all $n \in \mathbb{N}$.

Now let $(\xi_n)_{n \geqslant 1}$ be a sequence of independent, identically distributed real-valued random variables on $(\Omega, \mathfrak{A}, \mathsf{P})$ with $\mathsf{P}_{\xi_n} = N(0,1)$ for all $n \in \mathbb{N}$. Employing the notation $\lambda_n := \mathsf{P}_{\xi_n x_n}$ we obtain for all $a \in E'$ that

$$\hat{\lambda}_n(a) = \int e^{\mathrm{i}\langle \xi_n x_n, a \rangle} \, \mathrm{d}\mathsf{P} = \int e^{\mathrm{i}\langle \xi_n f_n, Ta \rangle} \, \mathrm{d}\mathsf{P}$$

$$= \int e^{\mathrm{i}r\langle f_n, Ta \rangle} \, N(0,1)(\mathrm{d}r)$$

$$= N(0,1)^\wedge(\langle f_n, Ta \rangle) = e^{-\frac{1}{2}\langle f_n, Ta \rangle^2}$$

for all $a \in E'$ and that λ_n is symmetric $(n \in \mathbb{N})$. For $\mu_n := \lambda_1 * \cdots * \lambda_n$ we have

$$\hat{\mu}_n(a) = \hat{\lambda}_1(a) \cdots \hat{\lambda}_n(a)$$

$$= \exp\left\{ -\frac{1}{2} \sum_{j=1}^{n} \langle f_j, Ta \rangle^2 \right\}$$

whenever $a \in E'$ $(n \in \mathbb{N})$. But then Parseval's identity yields

$$\lim_{n \to \infty} \hat{\mu}_n(a) = \exp \left\{ -\frac{1}{2} \sum_{j \geqslant 1} \langle f_j, Ta \rangle^2 \right\}$$

$$= \exp \left\{ -\frac{1}{2} \|Ta\|^2 \right\} = \hat{\rho}(a)$$

for all $a \in E'$, and from the Ito–Nisio theorem 3.1.10 together with the continuity theorem 2.1.9 the assertion follows.   □

3.2   Fourier expansion and construction of Brownian motion

As an application of the results of the previous section we shall return to the discussion of the Wiener measure introduced as a Gauss measure in Example 2.4.2.2 and its rise from Brownian motion in $\mathbb{R}$. In particular we shall establish the Fourier expansion of Brownian motion in $\mathbb{R}$ and construct Brownian motions in $\mathbb{R}$ with continuous paths.

In preparing the tools we recall the notion of p-**dimensional standard normal distribution** as the measure

$$N(0, I_p) := N(0, 1)^{\otimes p}$$

in $M^1(\mathbb{R}^p)$, where I_p denotes the p-dimensional unit matrix.

It is well-known that for any linear mapping T from $\mathbb{R}^q$ into $\mathbb{R}^p$ the Fourier representation

$$(T(N(0, I_q)))^\wedge(a) = \exp\left(-\frac{1}{2}\|T^\top a\|^2\right)$$
$$= \exp\left(-\frac{1}{2}\langle (TT^\top)a, a\rangle\right)$$

holds whenever $a \in \mathbb{R}^p$.

For any $p \times p$-matrix C over $\mathbb{R}$, $q \geqslant 1$ and any linear mapping T from $\mathbb{R}^q$ into $\mathbb{R}^p$ satisfying $TT^\top = C$ the measure

$$N(0, C) := T(N(0, I_q))$$

in $M^1(\mathbb{R}^p)$ turns out to be the (p-dimensional) **normal distribution with mean** (vector) 0 **and covariance** (matrix) C.

Remark 3.2.1

3.2.1.1 By the uniqueness of the Fourier transform (Theorem 2.1.4) the measure $N(0, C)$ is uniquely determined by its covariance C.

3.2.1.2 The covariance $C = TT^\top$ is obviously symmetric and positive semidefinite. Moreover, given a symmetric and positive semidefinite $p \times p$-matrix C over $\mathbb{R}$ there exists a $p \times p$-matrix R over $\mathbb{R}$ such that $C = RR^\top$, and consequently

$$N(0, C) = R(N(0, I_p))$$

or

$$N(0, C)^\wedge(a) = \exp\left(-\frac{1}{2}\langle Ca, a\rangle\right)$$

for all $a \in \mathbb{R}^p$.

3.2.1.3 Let $X = (X_1, \ldots, X_p)^\top$ be an $\mathbb{R}^p$-valued random variable on a probability space $(\Omega, \mathfrak{A}, \mathsf{P})$ with $\mathsf{P}_X = N(0, C)$. Then $\|X\|^2$ is P-integrable, and for the expectation vector and covariance (matrix) of X we obtain

$$\mathsf{E}(X) := (\mathsf{E}(X_1), \ldots, \mathsf{E}(X_p))^\top = 0$$

and

$$\mathsf{C}(X) := (\mathrm{Cov}(X_i, X_j))_{1 \leqslant i, j \leqslant p} = C$$

respectively.

The following

Properties 3.2.2 of the normal distribution will be used in the discussion below.

3.2.2.1 *With* $D(r_1, \ldots, r_p)$ *denoting the diagonal matrix containing* r_1, *. . . ,* $r_p \in \mathbb{R}$ *as diagonal elements we have*

$$N(0, D(\sigma_1^2, ..., \sigma_p^2)) = \bigotimes_{i=1}^{p} N(0, \sigma_i^2)$$

3.2.2.2 *Let* $X_1, \ldots, X_p$ *denote independent* $\mathbb{R}$*-valued random variables on* $(\Omega, \mathfrak{A}, \mathsf{P})$ *with* $\mathsf{P}_{X_i} = N(0, \sigma_i^2)$ *for all* $i = 1, \ldots, p$. *Then for* $X = (X_1, \ldots, X_p)^\top$ *one obtains*

$$\mathsf{P}_X = N(0, D(\sigma_1^2, \ldots, \sigma_p^2))$$

3.2.2.3 *Let* $X = (X_1, \ldots, X_p)^\top$ *be an* $\mathbb{R}^p$*-valued random variable on* $(\Omega, \mathfrak{A}, \mathsf{P})$ *with* $\mathsf{P}_X = N(0, C)$. *Then the real-valued random variables* $X_1, \ldots, X_p$ *are independent if and only if* $X_1, \ldots, X_p$ *are uncorrelated.*

3.2.2.4 *For any linear mapping* S *from* $\mathbb{R}^p$ *into* $\mathbb{R}^q$

$$S(N(0, C)) = N(0, SCS^\top)$$

3.2.2.5 *Let* $(C_n)_{n \geqslant 0}$ *be a sequence of symmetric and positive semidefinite* $p \times p$*-matrices over* $\mathbb{R}$. *Then the following statements are equivalent:*

(i) $N(0, C_n) \xrightarrow{\tau_w} N(0, C_0)$.
(ii) $C_n \to C_0$, *and the sequence* $(N(0, C_n))_{n \geqslant 1}$ *is* τ_w*-relatively compact* (*in* $M^1(\mathbb{R}^p)$).
(iii) $C_n \to C_0$.

While the proof of the implication (iii) $\Rightarrow$ (i) is an immediate consequence of the classical continuity theorem for measures on $\mathbb{R}^p$ the implication (i) $\Rightarrow$ (ii) makes use of the continuity theorem 2.1.9.

Now let $(\Omega, \mathfrak{A}, \mathsf{P}, (B_t)_{t \in I})$ denote a Brownian motion in $\mathbb{R}$ with parameter set $I = [0, 1]$. By $\mathcal{H}(B)$ we abbreviate the closed linear subspace of $L^2(\Omega, \mathfrak{A}, \mathsf{P})$ generated by the family $\{B_t : t \in I\}$ (of Brownian variables). It can be shown that $\mathcal{H}(B)$ is a separable Hilbert space with scalar product defined by

$$\langle \xi, \eta \rangle := \int \xi \eta \; \mathsf{d}\mathsf{P}$$

for all $\xi, \eta \in \mathcal{H}(B)$.

Theorem 3.2.3 *There is a unique linear isometry S form $L^2(I) := L^2(I, \mathfrak{B}(I), \lambda_I)$ onto $\mathcal{H}(B)$ satisfying*

$$S(\mathbb{1}_{[0,t]}) = B_t$$

for all $t \in I$.

Proof. At first let f be an elementary (or step) function on I of the form

$$f = \sum_{j=1}^n f_j \mathbb{1}_{[t_{j-1}, t_j[}$$

for a subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$ of I and coefficients $f_1, \ldots, f_n \in \mathbb{R}$.

Moreover, let

$$S(f) := \sum_{j=1}^n f_j (B_{t_j} - B_{t_{j-1}}) .$$

Then $f \mapsto S(f)$ is a linear mapping from the space $\mathcal{T}(I)$ of all elementary functions on I into the Hilbert space $\mathcal{H}(B)$ such that

$$S(\mathbb{1}_{[0,t]}) = B_t .$$

Therefore $S(\mathcal{T}(I))$ is dense in $\mathcal{H}(B)$. On the other hand $\mathcal{T}(I)$ is dense in $L^2(I)$.

Now, for every $f \in \mathcal{T}(I)$ we have that

$$\|S(f)\|_2 = \|f\|_2 .$$

Picking $f \in \mathcal{T}(I)$ of the form $f = \sum_{j=1}^{n} f_j \mathbb{1}_{[t_{j-1},t_j[}$ we see that $f^2 = \sum_{j=1}^{n} f_j^2 \mathbb{1}_{[t_{j-1},t_j[}$, and consequently

$$\int_{\Omega} S(f)^2 \, d\mathsf{P} = \sum_{i=1}^{n} \sum_{j=1}^{n} f_i f_j \int (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}}) \, d\mathsf{P}$$

$$= \sum_{j=1}^{n} f_j^2 (t_j - t_{j-1}) = \int_I f(t)^2 \, dt \,,$$

where for the latter equality independence and normal distribution of the increments of the given Brownian motion have been applied. Thus S is a linear isometry between the dense subspaces $\mathcal{T}(I)$ and $S(\mathcal{T}(I))$ of $L^2(I)$ and $\mathcal{H}(B)$ respectively, and hence extendable to a linear isometry S from $L^2(I)$ onto $\mathcal{H}(B)$.

Finally we note that the set $\{\mathbb{1}_{[0,t]} : t \in I\}$ is total in $L^2(I)$. As a consequence S is unique. $\qquad \square$

Definition 3.2.4 For every $f \in L^2(I)$ the unique element $S(f)$ of $L^2(\Omega, \mathfrak{A}, \mathsf{P})$ (constructed in Theorem 3.2.3) is called the **stochastic integral** of f with respect to the Brownian motion $(\Omega, \mathfrak{A}, \mathsf{P}, (B_t)_{t \in I})$ and is abbreviated by

$$\int_I f(t) \, dB_t$$

or

$$\int f \, dB \,.$$

Properties 3.2.5 (of the stochastic integral).

3.2.5.1 *For* $f, g \in L^2(I)$ *we have*

$$\mathsf{E}\left(\int f \, dB\right) = 0$$

and

$$\mathrm{Cov}\left(\int f \, dB, \int g \, dB\right) = \langle f, g \rangle \,.$$

In particular

$$\mathsf{V}\left(\int f \, dB\right) = \|f\|_2^2 \,.$$

For the proof we note that the mapping $\xi \mapsto \int \xi \, d\mathsf{P}$ from $L^2(\Omega, \mathfrak{A}, \mathsf{P})$ into $\mathbb{R}$ is continuous and identically zero on the dense subspace $S(\mathcal{T}(I))$ of

$\mathcal{H}(B)$. Therefore it vanishes on $\mathcal{H}(B)$. Now let $f, g \in L^2(I)$. Since $S(f)$ and $S(g)$ are centered and since S is an isometry we obtain that

$$\text{Cov}(S(f), S(g)) = \langle S(f), S(g) \rangle = \langle f, g \rangle$$

The remaining identity follows by choosing $g := f$.

3.2.5.2 *For any finite family* $\{f^{(i)} : i = 1, \ldots, k\}$ *of functions in* $L^2(I)$ *the* $\mathbb{R}^k$*-valued random variable*

$$\left(\int f^{(1)} \, dB, \ldots, \int f^{(k)} \, dB \right)^{\top}$$

has a (k-dimensional) normal distribution with mean 0 *and covariance* $(\langle f^{(i)}, f^{(j)} \rangle)_{1 \leqslant i, j \leqslant k}$.

Moreover, the real-valued random variables $\int f^{(1)} \, dB, \ldots, \int f^{(k)} \, dB$ *are independent if and only if the functions* $f^{(1)}, \ldots, f^{(k)} \in L^2(I)$ *are pairwise orthogonal.*

The second statement follows from the first one with the help of Property 3.2.2.3. For the proof of the first statement we first take functions $f^{(1)}, \ldots, f^{(k)} \in \mathcal{T}(I)$ of the form

$$f^{(i)} := \sum_{j=1}^{m} f_j^{(i)} \mathbb{1}_{[t_{j-1}, t_j[}$$

with $f_j^{(i)} \in \mathbb{R}$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ $(1 \leqslant i \leqslant k, \ 1 \leqslant j \leqslant m)$. By

$$R((x_j)_{1 \leqslant j \leqslant m}) := \left(\sum_{j=1}^{m} f_j^{(i)} x_j \right)_{1 \leqslant i \leqslant k}$$

for all $(x_j)_{1 \leqslant j \leqslant m} \in \mathbb{R}^m$ a linear mapping R from $\mathbb{R}^m$ into $\mathbb{R}^k$ is defined. From the properties of Brownian motion the distribution of the $\mathbb{R}^m$-valued random variable $X = (B_{t_j} - B_{t_{j-1}})_{1 \leqslant j \leqslant m}$ on $(\Omega, \mathfrak{A}, \mathsf{P})$ is $N(0, C)$ with $C := D(t_1 - t_0, \ldots, t_m - t_{m-1})$ (see Property 3.2.2.2). But now

$$(S(f^{(i)}))_{1 \leqslant i \leqslant k} = R \circ X,$$

hence by Property 3.2.2.4 $(S(f^{(i)}))_{1 \leqslant i \leqslant k}$ is normally distributed.

Next we pick $f^{(i)}, \ldots, f^{(k)} \in L^2(I)$ arbitrarily. There are sequences $(f_n^{(i)})_{n \geqslant 1}$ in $\mathcal{T}(I)$ such that $\lim_{n \to \infty} \|f_n^{(i)} - f^{(i)}\|_2 = 0$ whenever $i = 1, \ldots, k$. An application of Theorem 3.2.3 yields

$$\lim_{n \to \infty} \|S(f_n^{(i)}) - S(f^{(i)})\|_2 = 0.$$

Therefore the sequence $(S(f_n^{(i)}))_{n \geqslant 1}$ converges stochastically towards $S(f^{(i)})$ for all $i = 1, \ldots, k$, and consequently the sequence $((S(f_n^{(1)}), \ldots, S(f_n^{(k)})))_{n \geqslant 1}$ converges stochastically towards $(S(f^{(1)}), \ldots, S(f^{(k)}))$. Employing Application 1.2.15 we arrive at

$$\mathsf{P}_{(S(f_n^{(i)}))_{1 \leqslant i \leqslant k}} \xrightarrow{\ \tau_{\mathrm{w}}\ } \mathsf{P}_{(S(f^{(i)}))_{1 \leqslant i \leqslant k}} \, .$$

From the above discussion together with Property 3.2.5.1

$$\mathsf{P}_{(S(f_n^{(i)}))_{1 \leqslant i \leqslant k}} = N(0, C_n) \, ,$$

where $C_n := (\langle f_n^{(i)}, f_n^{(j)} \rangle)_{1 \leqslant i,j \leqslant k}$ (for all $n \geqslant 1$).

Let $C := (\langle f^{(i)}, f^{(j)} \rangle)_{1 \leqslant i,j \leqslant k}$. Then $\lim_{n \to \infty} \| f_n^{(i)} - f^{(i)} \|_2 = 0$ ($1 \leqslant i \leqslant k$) implies $\lim_{n \to \infty} C_n = C$. But now Property 3.2.2.5 yields

$$\mathsf{P}_{(S(f_n^{(i)}))_{1 \leqslant i \leqslant k}} \xrightarrow{\ \tau_{\mathrm{w}}\ } N(0, C) \, ,$$

hence

$$\mathsf{P}_{(S(f^{(i)}))_{1 \leqslant i \leqslant k}} = N(0, C)$$

which completes the proof.

Theorem 3.2.6 (Fourier expansion of Brownian motion)
Let $(f_n)_{n \geqslant 1}$ be an orthonormal basis of $L^2(I)$. For every $n \in \mathbb{N}$ let

$$\xi_n := \int f_n \, \mathrm{d}B \, .$$

Then $(\xi_n)_{n \geqslant 1}$ is a sequence of independent, identically distributed real-valued random variables on $(\Omega, \mathfrak{A}, \mathsf{P})$ with $\mathsf{P}_{\xi_n} = N(0,1)$ for every $n \in \mathbb{N}$, and

$$B_t = \sum_{n \geqslant 1} \left(\int_0^t f_n(u) \, \mathrm{d}u \right) \xi_n$$

uniformly in $t \in I$ P-a.s.

Proof. By Theorem 3.2.3 $(\xi_n)_{n \geqslant 1}$ is an orthonormal basis of $\mathcal{H}(B)$. Let

$$b_n(t) := \langle \mathbb{1}_{[0,t]}, f_n \rangle = \int_0^t f_n(u) \, \mathrm{d}u$$

for all $t \in I$ ($n \in \mathbb{N}$). Then

$$\mathbb{1}_{[0,t]} = \sum_{n \geqslant 1} b_n(t) f_n$$

in $L^2(I)$. Again employing Theorem 3.2.3 we arrive at

$$B_t = \int \mathbb{1}_{[0,t]} \, \mathrm{d}B$$

$$= \sum_{n \geqslant 1} b_n(t) \int f_n \, \mathrm{d}B$$

$$= \sum_{n \geqslant 1} b_n(t) \, \xi_n$$

in $L^2(\Omega, \mathfrak{A}, \mathsf{P})$. Now by Property 3.2.5.2 $\mathsf{P}_{\xi_n} = N(0,1)$ for all $n \in \mathbb{N}$, and the sequence $(\xi_n)_{n \geqslant 1}$ is independent. Since L^2-convergence implies stochastic convergence the equivalence theorem 3.1.6 yields that

$$B_t = \sum_{n \geqslant 1} \left(\int_0^t f_n(u) \, \mathrm{d}u \right) \xi_n \quad \mathsf{P}\text{-a.s.} \quad (t \in I).$$

In order to show the required *uniform* convergence P-a.s. we look at the sequence $(X_n)_{n \geqslant 1}$ of $C(I)$-valued random variables

$$\omega \mapsto \left(t \mapsto \left(\int_0^t f_n(u) \mathrm{d}u \right) \xi_n(\omega) \right)$$

on $(\Omega, \mathfrak{A}, \mathsf{P})$. Since $\mathsf{P}_{\xi_n} = N(0,1)$, P_{X_n} is symmetric for all $n \in \mathbb{N}$, and the sequence $(X_n)_{n \geqslant 1}$ is clearly independent.

Let $S_n := \sum_{i=1}^n X_i$ for every $\mathbb{N}$. Then our assertion reads as $S_n \xrightarrow{\mathsf{P}\text{-a.s.}} B$, where B is the $(\mathfrak{A} - \mathfrak{B}(C(I)))$-measurable) $C(I)$-valued random variable $\omega \mapsto (t \mapsto B_t(\omega))$ on $(\Omega, \mathfrak{A}, \mathsf{P})$. By the Ito–Nisio theorem 3.1.10 (and since $C(I)' = M^{\mathrm{b}}(I) - M^{\mathrm{b}}(I)$) it suffices to show that

$$\langle S_n, \mu \rangle \xrightarrow{\mathsf{P}\text{-stoch}} \langle B, \mu \rangle$$

for all $\mu \in M^{\mathrm{b}}(I)$, and for this limit relation it suffices in turn to show that

$$\lim_{n \to \infty} \int |\langle S_n, \mu \rangle - \langle B, \mu \rangle| \, \mathrm{d}\mathsf{P} = 0$$

whenever $\mu \in M^{\mathrm{b}}(I)$. Now we apply Fubini's theorem in order to obtain

$$\int |\langle S_n \mu \rangle - \langle B, \mu \rangle| \, \mathrm{d}\mathsf{P}$$

$$= \int_\Omega \left| \int_0^1 (S_n(\omega)(t) - B_t(\omega)) \, \mu(\mathrm{d}t) \right| \mathsf{P}(\mathrm{d}\omega)$$

$$\leqslant \int_0^1 \left(\int_\Omega |S_n(\omega)(t) - B_t(\omega)| \, \mathsf{P}(\mathrm{d}\omega) \right) \mu(\mathrm{d}t)$$

$$\leqslant \int_0^1 \left(\int_\Omega |S_n(\omega)(t) - B_t(\omega)|^2 \, \mathsf{P}(\mathrm{d}\omega) \right)^{1/2} \mu(\mathrm{d}t)$$

for all $\mu \in M^{\mathrm{b}}(I)$ $(n \in \mathbb{N})$. But

$$\int_\Omega [S_n(\omega)(t) - B_t(\omega)]^2 \, \mathsf{P}(\mathrm{d}\omega) = \sum_{k>n} b_k(t)^2$$

and

$$\sum_{k\geqslant 1} b_k(t)^2 = \sum_{k\geqslant 1} \langle \mathbb{1}_{[0,t]}, f_n \rangle^2 = \|\mathbb{1}_{[0,t]}\|_2^2 = t$$

for all $n \in \mathbb{N}$ $(t \in I)$. By the dominated convergence theorem this implies

$$\lim_{n\to\infty} \int_0^1 \left(\int_\Omega [S_n(\omega)(t) - B_t(\omega)]^2 \, \mathsf{P}(\mathrm{d}\omega) \right) \mu(\mathrm{d}t) = 0$$

for all $\mu \in M^{\mathrm{b}}(I)$, hence the assertion. $\qquad\square$

Remark 3.2.7 Employing the orthonormal basis $(f_n)_{n\geqslant 1}$ of $L^2(I)$ defined by

$$f_n(t) := \sqrt{2\sin(n\pi t)}$$

for which obviously

$$\int_0^t f_n(u)\mathrm{d}u = \sqrt{\frac{2}{n\pi}}\,(1 - \cos(n\pi t))$$

holds (for all $t \in I$, $n \geqslant 1$), Theorem 3.2.6 yields that

$$B_t = \sqrt{\frac{2}{n\pi}} \sum_{n\geqslant 1} (1 - \cos(n\pi t))\xi_n$$

holds uniformly in $t \in I$ P-a.s.

In the remaining part of this section we shall establish the existence of Brownian motion in $\mathbb{R}$ with continuous paths. In fact, the process $(B_t)_{t\in I}$ constructed in Remark 3.2.7 has continuous paths due to the uniform convergence of the series and the continuity of the functions f_n, $n \geqslant 1$.

Lemma 3.2.8 *For every* $n \in \mathbb{Z}_+$ *and* $k = 1, \ldots, 2^n$ *let*

$$h_n^{(k)} := \sqrt{2^n}\,\mathbb{1}_{[(2k-2)2^{-n-1},\,(2k-1)2^{-n-1}[}$$

$$- \sqrt{2^n}\,\mathbb{1}_{[(2k-1)2^{-n-1},\,(2k)2^{-n-1}[}\,.$$

Moreover, let $f_1 := \mathbb{1}_{[0,1]}$ *and*

$$f_{2^n+k} := h_n^{(k)}$$

for all $n \in \mathbb{Z}_+$, $k = 1, \ldots, 2^n$.

Then the so defined ***Haar system*** $(f_n)_{n\geqslant 1}$ *is an orthonormal basis of* $L^2(I)$.

Proof.   From

$$(h_n^{(k)})^2 = 2^n \mathbb{1}_{[(k-1)2^{-n}, k2^{-n}[}$$

follows $\|f_m\|_2 = 1$ for all $m \in \mathbb{N}$, and $f_m f_n = 0$ a.e. or $f_m f_n = c f_{m \wedge n}$ a.e. for all $m, n \in \mathbb{N}$ with $m \neq n$, where $c = c(m, n)$ denotes an appropriate constant, implies $\langle f_m, f_n \rangle = 0$ for $m \neq n$. Thus $(f_n)_{n \geqslant 1}$ is an orthonormal system in $L^2(I)$. In order to see that $(f_n)_{n \geqslant 1}$ is also a basis of $L^2(I)$ we pick $f \in L^2(I)$ with $\langle f, f_n \rangle = 0$ for all $n \in \mathbb{N}$ and set $F(t) := \int_0^t f(s) \mathrm{d}s$ for all $t \in I$. Clearly $F \in C(I)$. Moreover, by induction one shows that $F(k2^{-n}) = 0$ for all $n \in \mathbb{Z}_+$ and all $k = 0, 1, \ldots, 2^n$. Since F is continuous, we obtain

$$0 = F(t) = \int_0^t f(s) \, \mathrm{d}s$$

for all $t \in I$, hence that $f = 0$ a.e. $\square$

Lemma 3.2.9 *For every $t \in I$ and $m \geqslant 1$ let*

$$b_m(t) := \langle \mathbb{1}_{[0,t]}, f_m \rangle = \int_0^t f_m(s) \, \mathrm{d}s \, .$$

Then $b_m(t) \geqslant 0$ for all $t \in I$, $m \in \mathbb{N}$ and

$$\sum_{2^n < k \leqslant 2^{n+1}} b_k(t) \leqslant \frac{1}{2} 2^{-n/2}$$

whenever $t \in I$, $n \in \mathbb{Z}_+$.

Proof.   Evidently $b_m(t) \geqslant 0$ for all $t \in I$, $m \in \mathbb{N}$, and for $t \in [(j-1)2^{-n}, j2^{-n}[$ we have that

$$\sum_{2^n < k \leqslant 2^{n+1}} b_k(t) = b_j(t)$$

$$\leqslant b_j((2j-1)2^{-n-1})$$

$$= 2^{-(n+1)}\sqrt{2^n}$$

$$= \frac{1}{2} 2^{-n/2}$$

whenever $1 \leqslant j \leqslant 2^n$ $\square$

Lemma 3.2.10 *Let $(\xi_n)_{n \geqslant 1}$ be a sequence of independent, identically distributed real-valued random variables on a probability space $(\Omega, \mathfrak{A}, \mathsf{P})$ with $\mathsf{P}_{\xi_n} = N(0,1)$ for all $n \in \mathbb{N}$. Moreover, let $(X_n)_{n \geqslant 1}$ be the sequence of $C(I)$-valued random variables on $(\Omega, \mathfrak{A}, \mathsf{P})$ defined by*

$$X_n(\omega) := b_n(\cdot)\, \xi_n(\omega)$$

for all $\omega \in \Omega$, where the sequence $(b_n)_{n \geqslant 1}$ is given as in Lemma 3.2.9. Finally, let $S_n := \sum_{i=1}^{n} X_i$ for all $n \in \mathbb{N}$.

Then there exists a $C(I)$-valued random variable Z on $(\Omega, \mathfrak{A}, \mathsf{P})$ such that

$$S_n \xrightarrow{\text{P-a.s.}} Z \, .$$

Proof. At first we establish the inequalities

$$\mathsf{P}([|\xi_n| > u]) = \frac{2}{\sqrt{2\pi}} \int_u^\infty e^{-v^2/2} \, dv = \sqrt{\frac{2}{\pi}} \int_{u^2/2}^\infty (2r)^{-1/2} e^{-r} \, dr$$

$$= \sqrt{\frac{2}{\pi}} \Big[- e^{-r}(2r)^{-1/2} \Big]_{u^2/2}^\infty - \sqrt{\frac{2}{\pi}} \int_{u^2/2}^\infty (2r)^{-3/2} e^{-r} \, dr$$

$$\leqslant \sqrt{\frac{2}{\pi}} \frac{1}{u} e^{-u^2/2}$$

valid for all $u \in \mathbb{R}_+^\times$ $(n \in \mathbb{N})$. But then

$$\sum_{n \geqslant 2} \mathsf{P}\left(\Big[|\xi_n| > \sqrt{3 \ln n} \Big] \right) \leqslant \sqrt{\frac{2}{3\pi}} \sum_{n \geqslant 2} n^{-3/2} (\ln n)^{-1/2} < \infty \, .$$

An application of the Borel-Cantelli Lemma implies that

$$\mathsf{P}\left(\liminf_{n \geqslant 2} \Big[|\xi_n| \leqslant \sqrt{3 \ln n} \Big] \right) = 1 \, .$$

Therefore for P-almost every $\omega \in \Omega$ there exists an $n(\omega) \in \mathbb{N}$ such that $|\xi_m(\omega)| \leqslant \sqrt{3 \ln m}$ for all $m > 2^{n(\omega)}$, and hence we obtain

$$\alpha_n(\omega) := \max_{2^n < k \leqslant 2^{n+1}} |\xi_k(\omega)| \leqslant \sqrt{3 \ln 2^{n+1}} = (n+1)^{1/2} \sqrt{3 \ln 2}$$

whenever $n \geqslant n(\omega)$. But now Lemma 3.2.8 yields

$$\sum_{m > 2^{n(\omega)}} |b_m(t) \xi_m(\omega)| = \sum_{n \geqslant n(\omega)} \sum_{2^n < k \leqslant 2^{n+1}} b_k(t) |\xi_k(\omega)|$$

$$\leqslant \sum_{n \geqslant n(\omega)} \left(\sum_{2^n < k \leqslant 2^{n+1}} b_k(t) \right) \alpha_n(\omega)$$

$$\leqslant \sqrt{\frac{3 \ln 2}{2}} \sum_{n \geqslant n(\omega)} (n+1)^{1/2} 2^{-n/2} < \infty$$

for all $t \in I$ and P-almost all $\omega \in \Omega$. Consequently the series $\sum_{m \geqslant 1} b_m(t) \xi_m(\omega)$ converges uniformly in $t \in I$ for P-almost all $\omega \in \Omega$, and the proof is complete. $\qquad \square$

Theorem 3.2.11 (Existence of Brownian motion with continuous paths)
Let Z denote the $C(I)$-valued random variable on $(\Omega, \mathfrak{A}, \mathsf{P})$ established as the P-a.s. limit of the sequence $(S_n)_{n \geqslant 1}$ introduced in Lemma 3.2.10. For every $t \in I$ we consider the real-valued random variable Z_t defined on $(\Omega, \mathfrak{A}, \mathsf{P})$ by

$$Z_t(\omega) := \sum_{n \geqslant 1} S_n(t)(\omega) := \sum_{n \geqslant 1} b_n(t) \xi_n(\omega)$$

whenever $\omega \in \Omega$.

Then $(\Omega, \mathfrak{A}, \mathsf{P}, (Z_t)_{t \in I})$ is a Brownian motion on $\mathbb{R}$ having continuous paths.

Proof.   Since $Z(\omega) \in C(I)$ for all $\omega \in \Omega$, the process $(\Omega, \mathfrak{A}, \mathsf{P}, (Z_t)_{t \in I})$ has continuous paths.

Clearly, $Z_0(\omega) = 0$ for P-almost all $\omega \in \Omega$. It remains to show that the process $(\Omega, \mathfrak{A}, \mathsf{P}, (Z_t)_{t \in I})$ admits stationary independent increments $Z_t - Z_s$ with $N(0, t - s)$ as their distributions $(s, t \in I, \ s < t)$. Let $t_1, \ldots, t_k \in I$ with $t_1 < t_2 < \cdots < t_k$ be fixed for the sequel. For every $n \in \mathbb{N}$ let $T_n := (b_j(t_i))_{1 \leqslant i \leqslant k, 1 \leqslant j \leqslant n}$ and $\eta_n := (\xi_1, \ldots, \xi_n)^\top$. By Property 3.2.2.2 $\mathsf{P}_{\eta_n} = N(0, I_n)$, and by Property 3.2.2.4 we deduce from $(S_n(t_1), \ldots, S_n(t_k))^\top = T_n \eta_n$ that

$$\mathsf{P}_{(S_n(t_1), \ldots, S_n(t_k))} = N(0, T_n T_n^\top).$$

We now apply Lemma 3.2.10 in order to obtain that

$$\mathsf{P}_{(S_n(t_1), \ldots, S_n(t_k))} \xrightarrow{\tau_{\mathrm{w}}} \mathsf{P}_{(Z_{t_1}, \ldots, Z_{t_k})}.$$

On the other hand

$$\lim_{n \to \infty} T_n T_n^\top = (t_i \wedge t_j)_{1 \leqslant i, j \leqslant k} =: C$$

as follows easily from the equalities

$$\lim_{n \to \infty} (T_n T_n^\top)_{ij} = \lim_{n \to \infty} \sum_{m=1}^{n} b_m(t_i) b_m(t_j)$$

$$= \sum_{m \geqslant 1} \langle \mathbb{1}_{[0, t_i]}, f_m \rangle \langle \mathbb{1}_{[0, t_j]}, f_m \rangle$$

$$= \langle \mathbb{1}_{[0, t_i]}, \mathbb{1}_{[0, t_j]} \rangle = t_i \wedge t_j$$

valid for all $1 \leqslant i \leqslant k, \ 1 \leqslant j \leqslant n$.

But then Property 3.2.2.5 yields $\mathsf{P}_{(Z_{t_1}, \ldots, Z_{t_k})} = N(0, C)$. Introducing the linear mapping $R : \mathbb{R}^k \to \mathbb{R}^k$ given by

$$R(x_1, \ldots, x_k) := (x_1, x_2 - x_1, \ldots, x_k - x_{k-1})$$

for all $(x_1, \ldots, x_k) \in \mathbb{R}^k$ we see that $RCR^\top = D(t_1, t_2 - t_1, \ldots, t_k - t_{k-1})$, and once again applying Property 3.2.2.4 we obtain that

$$
\begin{aligned}
\mathsf{P}_{(Z_{t_1}, Z_{t_2} - Z_{t_1}, \ldots, Z_{t_k} - Z_{t_{k-1}})} &= R(N(0, C)) \\
&= N(0, D(t_1, t_2 - t_1, \ldots, t_k - t_{k-1})).
\end{aligned}
$$

Now, Properties 3.2.2.3 and 3.2.2.1 provide the final arguments.   □

3.3 Symmetric Lévy measures and generalized Poisson measures

In this section we shall extend the notion of Poisson measure by admitting not necessarily bounded though at least σ-finite exponents. Let $M^\sigma(E)$ denote the set of all σ-finite measures on $\mathfrak{B}(E)$.

Definition 3.3.1 A measure $\lambda \in M^\sigma(E)$ is said to be a **symmetric Lévy measure** if it has the following properties:

(a) λ is symmetric in the sense that $\lambda(-B) = \lambda(B)$ for all $B \in \mathfrak{B}(E)$.
(b) $\lambda(\{0\}) = 0$.
(c) There exists a measure $\tilde{e}(\lambda) \in M^b(E)$ such that

$$\widehat{\tilde{e}(\lambda)}(a) = \exp\left\{ \int (\cos\langle x, a \rangle - 1)\, \lambda(\mathrm{d}x) \right\}$$

for all $a \in E'$.

By $L_s(E)$ we abbreviate the totality of symmetric Lévy measures on E.

Remark 3.3.2

3.3.2.1 From the uniqueness of the Fourier transform (Theorem 2.1.4) follows that $\tilde{e}(\lambda)$ is uniquely determined by part (c) of its definition.

3.3.2.2 $\tilde{e}(\lambda)$ is a symmetric measure in $M^1(E)$ as one concludes from the facts that $\widehat{\tilde{e}(\lambda)}$ is real-valued and $\tilde{e}(\lambda)(E) = \widehat{\tilde{e}(\lambda)}(0) = 1$.

3.3.2.3 The correspondence $\lambda \mapsto \tilde{e}(\lambda)$ between the sets $L_s(E)$ and $\{\tilde{e}(\lambda) : \lambda \in L_s(E)\}$ is one-to-one.

  This assertion follows with some more effort as Theorem 3.3.11.

  We will also see later that at this stage

3.3.2.4 there is no need to assume that

$$\int (1 - \cos\langle x, y \rangle)\, \lambda(\mathrm{d}x) < \infty$$

or that $\widehat{\tilde{e}(\lambda)}(a) \neq 0$ for all $a \in E'$.

3.3.2.5 Let λ be a symmetric measure in $M^b(E)$ satisfying part (b) of Definition 3.3.1. Then $\lambda \in L_s(E)$, and $\tilde{e}(\lambda)$ is the Poisson measure $e(\lambda)$ with exponent λ (as defined in 2.4.8).

  This observation motivates the extension of Poisson measures to generalized Poisson measures to be introduced later.

3.3.2.6 For $\lambda_1, \lambda_2 \in L_s(E)$ the sum $\lambda_1 + \lambda_2 \in L_s(E)$, and

$$\tilde{e}(\lambda_1 + \lambda_2) = \tilde{e}(\lambda_1) * \tilde{e}(\lambda_2).$$

In fact, for every $a \in E'$

$$(\tilde{e}(\lambda_1) * \tilde{e}(\lambda_2))^\wedge(a) = \exp\left\{\int (\cos\langle x, y\rangle - 1)(\lambda_1 + \lambda_2)(\mathrm{d}x)\right\},$$

hence by the uniqueness of the Fourier transform (Theorem 2.1.4) the assertion follows.

We now turn our attention to the study of sequences in the set $P(E)$ of Poisson measures on E.

For $\lambda \in M^\sigma(E)$ and $\delta \in \mathbb{R}_+^\times$ we introduce the abbreviations

$$\lambda|_\delta := \lambda_{U_\delta}$$

and

$$\lambda|^\delta := \lambda_{\complement U_\delta},$$

where $U_\delta := \{x \in E : \|x\| \leqslant \delta\}$. Clearly, $\lambda|_\delta + \lambda|^\delta = \lambda$. Moreover, let

$$C(\lambda) := \{\delta \in \mathbb{R}_+^\times : \lambda(\partial U_\delta) = 0\}$$

Evidently $\complement C(\lambda)$ is a countable set.

Theorem 3.3.3 *Let* $(\lambda_n)_{n \geqslant 1}$ *be a sequence in* $M^b(E)$ *such that* $(e(\lambda_n))_{n \geqslant 1}$ *is relatively shift compact. Then for every* $\delta \in \mathbb{R}_+^\times$ *the sequence* $(\lambda_n|^\delta)_{n \geqslant 1}$ *is* τ_w-*relatively compact.*

Proof. 1. For $\delta > 0$ we have that $\lambda_n|_\delta + \lambda_n|^\delta = \lambda_n$, hence that $e(\lambda_n|^\delta) \prec e(\lambda_n)$. But then Property 2.2.16.5 yields that $(e(\lambda_n|^\delta))_{n \geqslant 1}$ is relatively shift compact. As a consequence we may assume without loss of generality that $\lambda_n(U_\delta) = 0$, hence that $\lambda_n|^\delta = \lambda_n$ for all $n \in \mathbb{N}$. Moreover, by Theorem 2.2.25 $(|e(\lambda_n)|^2)_{n \geqslant 1}$ is τ_w-relatively compact, and from Property 2.4.10.6 we obtain $|e(\lambda_n)|^2 = e(\lambda_n + \lambda_n^-)$ whenever $n \in \mathbb{N}$. Therefore, in view of Theorem 2.2.22 we may assume without loss of generality that λ_n is symmetric for all $n \in \mathbb{N}$.

2. We show that $d := \sup_{n \geqslant 1} \|\lambda_n\| < \infty$.

Suppose that $d = \infty$. By an eventual choice of an appropriate subsequence we may assume without loss of generality that $\|\lambda_n\| \geqslant n$ for all $n \in \mathbb{N}$ and that

$$e(\lambda_n) \xrightarrow{\tau_w} \mu \in M^1(E)$$

(see part 1. of this proof). For every $n \in \mathbb{N}$ we now consider the measure $\sigma_n := \frac{1}{\|\lambda_n\|}\lambda_n$ which is symmetric and satisfies $n\sigma_n \leqslant \lambda_n$ as well as $\sigma_n(U_\delta) = 0$. Moreover, we have that $e(\sigma_n)^n \prec e(\lambda_n)$ for all $n \in \mathbb{N}$.

Applying Property 2.2.16.5 and Theorem 2.2.19 yields the τ_w-relative compactness of $(\mathrm{e}(\sigma_n))_{n\geqslant 1}$. Let ν be an accumulation point of $(\mathrm{e}(\sigma_n))_{n\geqslant 1}$. From

$$\mathrm{e}(\sigma_n)^m \prec \mathrm{e}(\lambda_n)$$

for all $n \geqslant m$ follows that $\nu^m \prec \mu$ for all $m \in \mathbb{N}$ and hence that $(\nu^m)_{m\geqslant 1}$ is τ_w-relatively compact. We now infer from Lemma 2.2.1 that $\nu = \varepsilon_0$ and thus that

$$\mathrm{e}(\sigma_n) \xrightarrow{\tau_\mathrm{w}} \varepsilon_0\,.$$

Since $0 \notin \partial(\complement U_\delta)$, $\complement U_\delta$ is an ε_0-continuity set, and the desired contradiction follows from the inequalities

$$0 = \varepsilon_0\big(\complement U_\delta\big) = \lim_{n\to\infty} \mathrm{e}(\sigma_n)\big(\complement U_\delta\big)$$

$$\geqslant \frac{1}{\mathrm{e}} \limsup_{n\geqslant 1} \sigma_n\big(\complement U_\delta\big) = \frac{1}{\mathrm{e}}\,.$$

3. Finally, let $\varepsilon > 0$. From part 1. of this proof we conclude that there exists a set $K \in \mathcal{K}(E)$ such that $\mathrm{e}(\lambda_n)\big(\complement K\big) \leqslant \varepsilon$ for all $n \in \mathbb{N}$. Now we apply part 2. of this proof in order to arrive at the estimate

$$\lambda_n\big(\complement K\big) \leqslant \mathrm{e}^{\|\lambda_n\|}\mathrm{e}(\lambda_n)\big(\complement K\big) \leqslant \mathrm{e}^d\varepsilon$$

valid for all $n \in \mathbb{N}$. But then $(\lambda_n)_{n\geqslant 1}$ is a τ_w-relatively compact sequence by Prohorov's theorem 1.3.7. $\qquad\square$

Theorem 3.3.4 *Let* $(\lambda_n)_{n\geqslant 1}$ *be a sequence in* $M^\mathrm{b}(E)$ *such that* $(\mathrm{e}(\lambda_n))_{n\geqslant 1}$ *is relatively shift compact. For every* $n \in \mathbb{N}$ *let*

$$f_n(a) := \int_{U_1} \langle x, a\rangle^2\, \lambda_n(\mathrm{d}x)$$

whenever $a \in V_1$. *Then* $(f_n)_{n\geqslant 1}$ *is relatively compact in* $C(V_1)$.

Proof. Replacing λ_n by $\lambda_n + \lambda_n^-$ changes f_n by the factor 2. Therefore we may assume without loss of generality that λ_n is symmetric and that $(\mathrm{e}(\lambda_n))_{n\geqslant 1}$ is τ_w-relatively compact. This can be justified by referring to Property 2.4.10.6 and Theorem 2.2.25 respectively. Thus, with the help of Theorem 2.1.8, we obtain that $(\mathrm{e}(\lambda_n)^\wedge)_{n\geqslant 1}$ is relatively compact in $C(V_1)$. But now we observe that $1 - \cos t \geqslant t^2/3$ for all $t \in [-1, 1]$. It follows that

$$f_n(a) \leqslant -3\int \left(\cos\langle x, a\rangle - 1\right) \lambda_n(\mathrm{d}x) = -3\,\mathrm{Log}\,\mathrm{e}(\lambda_n)^\wedge(a)$$

for all $a \in E'$. Moreover, $\mathrm{e}(\lambda_n)^\wedge(a) > 0$ for all $n \in \mathbb{N}$. Thus the relative compactness of $(\mathrm{e}(\lambda_n)^\wedge)_{n \geqslant 1}$ in $C(V_1)$ implies that

$$\inf_{n \geqslant 1} \mathrm{e}(\lambda_n)^\wedge(a) > 0$$

for all $a \in V_1$.

In fact, for some subsequence $(\lambda_{n_k})_{k \geqslant 1}$ of $(\lambda_n)_{n \geqslant 1}$ we have that

$$\mathrm{e}(\lambda_{n_k}) \xrightarrow{\ \tau_{\mathrm{w}}\ } \mu \in \mathrm{I}(E)$$

where $\hat{\mu}(a) \neq 0$ for all $a \in E'$.

But then $\sup_{n \geqslant 1} |f_n(a)| < \infty$ for all $a \in V_1$. On the other hand we know that the set $\{f_n : n \in \mathbb{N}\}$ is equicontinuous in 0 (with respect to $\tau(E', E)$).

This is easily deduced from Property 2.1.6.5 together with the fact that $\mathrm{e}(\lambda_n)^\wedge(0) = 1$ for all $n \in \mathbb{N}$.

Finally,

$$|f_n(a)^{1/2} - f_n(b)^{1/2}| \leqslant f_n(a - b)$$

for all $a, b \in V_1$ satisfying $a - b \in V_1$ (and all $n \in \mathbb{N}$).

In fact, for such $a, b \in V_1$

$$\int_{U_1} \langle x, a \rangle \langle x, b \rangle \, \lambda_n(\mathrm{d}x) \leqslant f_n(a)^{1/2} f_n(b)^{1/2}$$

hence

$$
\begin{aligned}
|f_n(a)^{1/2} &- f_n(b)^{1/2}|^2 \\
&\leqslant f_n(a) + f_n(b) - 2 f_n(a)^{1/2} f_n(b)^{1/2} \\
&\leqslant \int_{U_1} \langle x, a \rangle^2 \, \lambda_n(\mathrm{d}x) + \int_{U_1} \langle x, b \rangle^2 \, \lambda_n(\mathrm{d}x) - 2 \int_{U_1} \langle x, a \rangle \langle x, b \rangle \, \lambda_n(\mathrm{d}x) \\
&= \int_{U_1} (\langle x, y \rangle - \langle x, b \rangle)^2 \, \lambda_n(\mathrm{d}x) \\
&= \int_{U_1} \langle x, a - b \rangle^2 \, \lambda_n(\mathrm{d}x) = f_n(a - b) \,.
\end{aligned}
$$

From the inequality just established we infer that $\{f_n : n \in \mathbb{N}\}$ is equicontinuous everywhere on V_1 (with respect to $\tau(E', E)$). The Arzelà–Ascoli theorem yields the assertion. $\qquad\square$

Theorem 3.3.5 *Let $(\lambda_n)_{n \geqslant 1}$ be a sequence of measures in $\mathrm{L_s}(E) \cap M^{\mathrm{b}}(E)$ such that $\lambda_n \uparrow \lambda \in \mathrm{L_s}(E)$. Then*

$$\tilde{\mathrm{e}}(\lambda_n) = \mathrm{e}(\lambda_n) \xrightarrow{\ \tau_{\mathrm{w}}\ } \tilde{\mathrm{e}}(\lambda) \,.$$

Proof.   By Property 2.4.10.4 we have that $\tilde{e}(\lambda_n) \prec \tilde{e}(\lambda_{n+1})$ for all $n \in \mathbb{N}$. Since $\lambda_n \leqslant \lambda$, the Radon–Nikodym theorem provides a measurable function $f_n : E \to I$ such that $\lambda_n = f_n \cdot \lambda$ $(n \in \mathbb{N})$. From $\lambda_n \uparrow \lambda$ we infer that $f_n \uparrow \mathbb{1}_E$ λ-a.e. Now the monotone convergence theorem applies and we obtain that

$$\lim_{n\to\infty} \int (1 - \cos\langle x, a\rangle)\, \lambda_n(\mathrm{d}x)$$

$$= \lim_{n\to\infty} \int (1 - \cos\langle x, a\rangle) f_n(x)\, \lambda(\mathrm{d}x)$$

$$= \int (1 - \cos\langle x, a\rangle)\, \lambda(\mathrm{d}x)\,,$$

hence that

$$\lim_{n\to\infty} \tilde{e}(\lambda_n)^{\wedge}(a) = \tilde{e}(\lambda)^{\wedge}(a)$$

for all $a \in E'$. But every measure $\tilde{e}(\lambda_n)$ is symmetric $(n \in \mathbb{N})$. Thus, by the Ito–Nisio theorem 3.1.10 the assertion follows. $\square$

Properties 3.3.6 of symmetric Lévy measures.
Let $\lambda \in \mathrm{L_s}(E)$.
3.3.6.1 *For every $\delta \in \mathbb{R}_+^{\times}$ we have $\lambda(\complement U_\delta) < \infty$.*
     In fact, λ being σ-finite there exists a sequence $(\lambda_n)_{n\geqslant 1}$ of symmetric measures in $M^{\mathrm{b}}(E)$ with $\lambda_n \uparrow \lambda$. But then Theorem 3.3.5 implies

$$\tilde{e}(\lambda_n) \xrightarrow{\tau_{\mathrm{w}}} \tilde{e}(\lambda)\,,$$

and hence Theorem 3.3.3 that

$$\lambda(\complement U_\delta) = \sup_{n\geqslant 1} \lambda_n(\complement U_\delta) = \sup_{n\geqslant 1}(\lambda_n|^\delta)(E) < \infty\,.$$

3.3.6.2 *For every sequence $(\delta_n)_{n\geqslant 1}$ in $\mathbb{R}_+^{\times}$ with $\delta_n \downarrow 0$ we have that*

$$\tilde{e}(\lambda|^{\delta_n}) \xrightarrow{\tau_{\mathrm{w}}} \tilde{e}(\lambda)$$

For a proof we note that from Property 3.3.6.1 together with Remark 3.3.2.5 follows that $\lambda_n := \lambda|^{\delta_n} \in \mathrm{L_s}(E) \cap M^{\mathrm{b}}(E)$ for every $n \in \mathbb{N}$. Evidently $\lambda_n \uparrow \lambda$, so that Theorem 3.3.5 yields the assertion.

3.3.6.3

$$\sup\left\{ \int_{U_1} \langle x, a\rangle^2\, \lambda(\mathrm{d}x) : a \in V_1 \right\} < \infty$$

In order to see this let $(\delta_n)_{n \geqslant 1}$ be a sequence in $\mathbb{R}_+^\times$ with $\delta_n \downarrow 0$ and let $\lambda_n := \lambda|^{\delta_n}$ for all $n \in \mathbb{N}$. Then $\lambda_n \in M^b(E)$ for all $n \in \mathbb{N}$ by Property 3.3.6.1 and

$$\tilde{e}(\lambda_n) \xrightarrow{\ \tau_w\ } \tilde{e}(\lambda)$$

by Property 3.3.6.2. But then by Theorem 3.3.4 there exists an $\alpha \in \mathbb{R}_+^\times$ satisfying

$$\int_{U_1} \langle x, a \rangle^2 \, \lambda_n(\mathrm{d}x) \leqslant \alpha$$

for all $a \in V_1$, $n \in \mathbb{N}$, and since $\lambda_n \uparrow \lambda$ we obtain that

$$\int_{U_1} \langle x, a \rangle^2 \, \lambda(\mathrm{d}x) \leqslant \alpha$$

(appealing to the proof of Theorem 3.3.5).

3.3.6.4 *(See Remark 3.3.2.4)*

$$\int (1 - \cos\langle x, a \rangle) \, \lambda(\mathrm{d}x) < \infty$$

and

$$\lim_{t \to \infty} \int \frac{1 - \cos t\langle x, a \rangle}{t^2} \, \lambda(\mathrm{d}x) = 0$$

for all $a \in E'$.

For the proof let, given $a \in E'$,

$$f_a(x) := \langle x, a \rangle^2 \, \mathbb{1}_{U_1}(x) + 2 \cdot \mathbb{1}_{\complement U_1}(x)$$

whenever $x \in E$. From Properties 3.3.6.1 and 3.3.6.3 we infer that f_a is λ-integrable. Since $1 - \cos s \leqslant \frac{s^2}{2}$ for all $s \in \mathbb{R}$ we have that

$$\frac{1 - \cos t\langle x, a \rangle}{t^2} \leqslant f_a(x)$$

for all $x \in E$, $t \geqslant 1$. The special choice $t = 1$ yields the first assertion. The remaining one is implied by the limit relation

$$\lim_{t \to \infty} \frac{1 - \cos t\langle x, a \rangle}{t^2} = 0$$

valid for all $x \in E$ with the help of Lebesgue's dominated convergence theorem.

Theorem 3.3.7 (Characterization of symmetric Lévy measures)
Let λ be a symmetric measure in $M^\sigma(E)$ with $\lambda(\{0\}) = 0$. Then the following statements are equivalent:

(i) $\lambda \in L_s(E)$.

(ii) *For each $\delta \in \mathbb{R}_+^\times$ we have that $\lambda(\complement U_\delta) < \infty$, and for some (each) sequence $(\delta_n)_{n \geqslant 1}$ in $\mathbb{R}_+^\times$ with $\delta_n \downarrow 0$ the sequence $(\tilde{e}(\lambda|^{\delta_n}))_{n \geqslant 1}$ is τ_w-relatively compact.*

(iii) *There exists a sequence $(\lambda_n)_{n \geqslant 1}$ of symmetric measures in $M^b(E)$ with $\lambda_n \uparrow \lambda$ such that the sequence $(\tilde{e}(\lambda_n))_{n \geqslant 1}$ is τ_w-relatively compact.*

Proof. (i) $\Rightarrow$ (ii) follows directly from Properties 3.3.6.1 and 3.3.6.2.

(ii) $\Rightarrow$ (iii). The choice $\lambda_n := \lambda|^{\delta_n}$ for all $n \in \mathbb{N}$ yields the implication.

(iii) $\Rightarrow$ (i). The defining properties (a) and (b) of Definition 3.3.1 of a symmetric Lévy measure are part of the assumptions. From $\lambda_n \uparrow \lambda$ we obtain that for all $a \in E'$

$$\lim_{n \to \infty} \int (1 - \cos\langle x, a \rangle) \, \lambda_n(dx) = \int (1 - \cos\langle x, a \rangle) \, \lambda(dx)$$

(see the proof of Theorem 3.3.5). Then Properties 3.3.6.1 and 3.3.6.4 imply that

$$\lim_{n \to \infty} \tilde{e}(\lambda_n)^\wedge(a) = \exp\left\{ \int (\cos\langle x, a \rangle - 1) \, \lambda(dx) \right\}$$

for all $a \in E'$. From the continuity theorem follows property (c) of the Definition 3.3.1 and hence (i). $\qquad\square$

Corollary 3.3.8 *Suppose that $\lambda \in L_s(E)$.*

(i) *Let σ be a symmetric measure on E with $\sigma \leqslant \lambda$. Then $\sigma \in L_s(E)$, and*

$$\tilde{e}(\sigma) \prec \tilde{e}(\lambda) \,.$$

(ii) $\tilde{e}(\lambda) \in I(E)$.

Moreover, $\tilde{e}(\lambda)$ is (continuously) embeddable ($\in EM(E)$) with (continuous) $(\tilde{e}(\lambda)_t)_{t \geqslant 0}$, where

$$\tilde{e}(\lambda)_t := \tilde{e}(t\lambda)$$

for all $t \in \mathbb{R}$.

Proof. (i). Obviously the measure σ belongs to $M^\sigma(E)$ and fulfills properties (a) and (b) of the Lévy measure. For every $\delta \in \mathbb{R}_+^\times$ we deduce from Property 3.3.6.1 that $\sigma(\complement U_\delta) \leqslant \lambda(\complement U_\delta) < \infty$. Now let $(\delta_n)_{n \geqslant 1}$ be a sequence in $\mathbb{R}_+^\times$ with $\delta_n \downarrow 0$. Then for every $n \in \mathbb{N}$ we obtain that

$$\tilde{e}(\sigma|^{\delta_n}) * \tilde{e}((\lambda - \sigma)|^{\delta_n}) = \tilde{e}(\lambda|^{\delta_n})$$

holds and that $\tilde{e}(\sigma|^{\delta_n})$ is symmetric. (See Properties 2.4.10.2 and 2.4.10.5 and observe that $0 \leqslant \lambda - \sigma \leqslant \lambda$.) From Property 3.3.6.2 and Theorem 2.2.21 we infer that the sequence $(e(\sigma|^{\delta_n}))_{n \geqslant 1}$ is τ_{w}-relatively compact. Thus Theorem 3.3.7 implies the assertion.

(ii). In view of Remark 3.3.2.6 and (i) of this corollary we have $t\lambda \in L_s(E)$ and

$$\tilde{e}(s\lambda) * \tilde{e}(t\lambda) = \tilde{e}((s+t)\lambda)$$

for all $s, t \in \mathbb{R}_+$. This shows that $\tilde{e}(\lambda) \in I(E)$. The remaining part of the statement follows from

$$\mathrm{Log}\,\tilde{e}(\lambda)^\wedge(a) = \int (\cos\langle x, a\rangle - 1)\,\lambda(\mathrm{d}x)$$

valid for all $a \in E'$ and from the embedding theorem 2.3.9. (Here we rely on Properties 2.1.6.4, 3.3.6.4 and on Theorem 2.1.10 for detailed arguments.) $\qquad\square$

Definition 3.3.9 For $\lambda \in L_s(E)$ the measure $\tilde{e}(\lambda) \in M^1(E)$ introduced in Definition 3.3.1 is called the **generalized Poisson measure with exponent** λ.

Clearly,

$$\mathrm{Log}\,\tilde{e}(\lambda)^\wedge(a) = \int (\cos\langle x, a\rangle - 1)\,\lambda(\mathrm{d}x)$$

for every $a \in E'$.

Lemma 3.3.10 Let $\lambda_1, \lambda_2 \in L_s(E)$ satisfying

$$\int_B (1 - \cos\langle x, a\rangle)\,\lambda_1(\mathrm{d}x) = \int_B (1 - \cos\langle x, a\rangle)\,\lambda_2(\mathrm{d}x)$$

for all $B \in \mathfrak{B}(E)$ and $a \in E'$. Then $\lambda_1 = \lambda_2$.

Proof. Remark 3.3.2.1 and property (b) of the Lévy measure together with the fact that $\complement U_{1/n} \uparrow E \setminus \{0\}$ holds enable us to assume without loss of generality that λ_1 and λ_2 belong to $M^{\mathrm{b}}(E)$.

Now we define for every $a \in E'$

$$g_a(x) := \sum_{m \geqslant 1} \frac{1}{2^{m+1}} \left(1 - \cos\left(\frac{\langle x, a\rangle}{m}\right)\right)$$

whenever $x \in E$. Clearly $g_a \in C(E)$, $0 \leqslant g_a \leqslant 1$, and $g_a(x) = 0$ if and only if $\langle x, a\rangle = 0$ $(x \in E)$. From Appendix B.5 (Banach, Hahn) we

deduce the existence of a sequence $(a_j)_{j\geqslant 1}$ in E' with the property that $\langle x, a_j \rangle = 0$ for all $j \in \mathbb{N}$ implies that $x = 0$. In fact, for every sequence $(x_n)_{n\geqslant 1}$ dense in E there exists a sequence $(a_n)_{n\geqslant 1}$ in E' satisfying $\|a_n\| = 1$ and $\langle x, a_n \rangle = \|x_n\|$ for all $n \in \mathbb{N}$.

Now let

$$h(x) := \sum_{j \geqslant 1} \frac{1}{2^j} g_{a_j}(x)$$

for all $x \in E$. Then $h \in C(E)$, $0 \leqslant h \leqslant 1$, and $h(x) = 0$ if and only if $x = 0$ $(x \in E)$. By assumption

$$\int_B h \, \mathrm{d}\lambda_1 = \int_B h \, \mathrm{d}\lambda_2$$

for all $B \in \mathfrak{B}(E)$, hence $\lambda_1(B) = \lambda_2(B)$ for all $B \in \mathfrak{B}(E)$ with $0 \notin B$. Property (b) of a Lévy measure leads to the assertion. □

Theorem 3.3.11 (Injectivity of the generalized Poisson mapping; see Remark 3.3.2.3)

Let $\lambda_1, \lambda_2 \in \mathrm{L_s}(E)$ with $\tilde{\mathrm{e}}(\lambda_1) = \tilde{\mathrm{e}}(\lambda_2)$. Then $\lambda_1 = \lambda_2$.

Proof. For each $a \in E'$ let

$$f_a(x) := 1 - \cos\langle x, a \rangle$$

whenever $x \in E$. By assumption we have that

$$\exp\left(-\int f_a \, \mathrm{d}\lambda_1\right) = \tilde{\mathrm{e}}(\lambda_1)^{\wedge}(a)$$

$$= \tilde{\mathrm{e}}(\lambda_2)^{\wedge}(a)$$

$$= \exp\left(-\int f_a \, \mathrm{d}\lambda_2\right),$$

hence after an application of Property 3.3.6.4 (of Lévy measures) that

$$\int f_a \, \mathrm{d}\lambda_1 = \int f_a \, \mathrm{d}\lambda_2$$

for all $a \in E'$. We note that

$$\mathrm{e}^{\mathrm{i}(\alpha+\beta)} + \mathrm{e}^{\mathrm{i}(\alpha-\beta)} = \mathrm{e}^{\mathrm{i}\alpha}(\mathrm{e}^{\mathrm{i}\beta} + \mathrm{e}^{-\mathrm{i}\beta}) = 2\mathrm{e}^{\mathrm{i}\alpha}\cos\beta$$

(for $\alpha, \beta \in \mathbb{R}$), hence that

$$f_{a+b} + f_{a-b} - 2f_a = 2\cos\langle \cdot, a \rangle f_b$$

for all $a, b \in E'$. But then

$$
\begin{aligned}
(f_b \cdot \lambda_1)^\wedge(a) &= \int \cos\langle \cdot, a \rangle f_b \, d\lambda_1 \\
&= \int \cos\langle \cdot, a \rangle f_b \, d\lambda_2 \\
&= (f_b \cdot \lambda_2)^\wedge(a)
\end{aligned}
$$

for all $a, b \in E'$. The uniqueness theorem 2.1.4 yields $f_b \cdot \lambda^1 = f_b \cdot \lambda^2$ for all $b \in E'$, and Lemma 3.3.10 completes the proof. $\qquad \square$

Theorem 3.3.12 *Let $(\lambda_n)_{n \geqslant 1}$ be a sequence in $\mathrm{L}_{\mathrm{s}}(E)$ such that the corresponding sequence $(\tilde{\mathrm{e}}(\lambda_n))_{n \geqslant 1}$ of generalized Poisson measures is τ_{w}-relatively compact in $M^1(E)$. Then*

(i) *for every $\delta \in \mathbb{R}_+^\times$ the sequence $(\lambda_n|^\delta)_{n \geqslant 1}$ is τ_{w}-relatively compact in $M^{\mathrm{b}}(E)$,*

(ii) *there exist a subsequence $(\lambda_{n_k})_{k \geqslant 1}$ of $(\lambda_n)_{n \geqslant 1}$ and a $\lambda \in \mathrm{L}_{\mathrm{s}}(E)$ such that*

$$
\lambda_{n_k}|^\delta \xrightarrow{\ \tau_{\mathrm{w}}\ } \lambda|^\delta
$$

for all $\delta \in C(\lambda)$, and

(iii) *the sequence $(f_n)_{n \geqslant 1}$ defined by*

$$
f_n(a) := \int_{U_1} \langle x, a \rangle^2 \, \lambda_n(dx)
$$

for all $a \in V_1$, $n \in \mathbb{N}$, is relatively compact in $C(V_1)$.

Proof. (i). From Corollary 3.3.8 (i) we infer that $\tilde{\mathrm{e}}(\lambda_n|^\delta) \prec \tilde{\mathrm{e}}(\lambda_n)$ for all $n \in \mathbb{N}$. Consequently, by Theorem 2.2.21 the sequence $(\tilde{\mathrm{e}}(\lambda_n|^\delta))_{n \geqslant 1}$ is τ_{w}-relatively compact. Property 3.3.6.1 yields that $\lambda_n|^\delta \in M^{\mathrm{b}}(E)$ for all $n \in \mathbb{N}$. The desired assertion now follows from Theorem 3.3.3.

(ii). The assumption together with part (i) of this proof supply us with a subsequence $(\lambda_{n_k})_{k \geqslant 1}$ of $(\lambda_n)_{n \geqslant 1}$ satisfying

$$
\tilde{\mathrm{e}}(\lambda_{n_k}) \xrightarrow{\ \tau_{\mathrm{w}}\ } \mu \in M^1(E)
$$

and

$$
\lambda_{n_k}|^{1/j} \xrightarrow{\ \tau_{\mathrm{w}}\ } \lambda^{(j)} \in M^{\mathrm{b}}(E)
$$

for all $j \in \mathbb{N}$. (Here the usual diagonal procedure has been applied.) The set $C := \bigcap_{j \geqslant 1} C(\lambda^{(j)})$ has a countable complement in $\mathbb{R}_+^\times$. Let $j, m \in \mathbb{N}$ and $\delta \in C$ with $\delta > \frac{1}{j}$ and $\delta > \frac{1}{m}$. By Corollary 1.2.10 we obtain that

$$
\lambda^{(j)}|^\delta = \lambda^{(m)}|^\delta .
$$

Now let $(\delta_j)_{j \geqslant 1}$ be a sequence in C with $\delta_j \downarrow 0$ and $\delta_j > \frac{1}{j}$ for all $j \in \mathbb{N}$. We put

$$\lambda(B) := \sup_{j \geqslant 1} (\lambda^{(j)}|^{\delta_j})(B)$$

for all $B \in \mathfrak{B}(E)$. Observing that $\lambda^{(j)}|^{\delta_j} = \lambda^{(j+1)}|^{\delta_j} \leqslant \lambda^{(j+1)}|^{\delta_{j+1}}$ for all $j \in \mathbb{N}$ we see that λ is a symmetric measure in $M^{\sigma}(E)$ with $\lambda(\{0\}) = 0$. Moreover, for given $\delta \in C(\lambda)$ we choose $j \in \mathbb{N}$ with $\delta_j < \delta$. Then $\delta \in C(\lambda^{(j)})$, and from Corollary 1.2.10 we deduce that

$$\lambda_{n_k}|^{\delta} = (\lambda_{n_k}|^{1/j})^{\delta} \xrightarrow{\tau_{\mathrm{w}}} \lambda^{(j)}|^{\delta} = (\lambda^{(j)}|^{\delta_j})^{\delta} = (\lambda|^{\delta_j})^{\delta} = \lambda|^{\delta} \,,$$

where $\lambda|^{\delta} \in M^{\mathrm{b}}(E)$. Thus, it remains to show that $\lambda \in \mathrm{L}_{\mathrm{s}}(E)$. For this we first note that by Theorem 2.4.12 (ii) we have that

$$\tilde{\mathrm{e}}(\lambda_{n_k}|^{\delta_j}) \xrightarrow{\tau_{\mathrm{w}}} \tilde{\mathrm{e}}(\lambda|^{\delta_j})$$

for all $j \in \mathbb{N}$. Next we employ Corollary 3.3.8 (i) and Remark 3.3.2 (6) in order to obtain

$$\tilde{\mathrm{e}}(\lambda_{n_k}) = \tilde{\mathrm{e}}(\lambda_{n_k}|_{\delta_j}) * \tilde{\mathrm{e}}(\lambda_{n_k}|^{\delta_j})$$

valid for all $i, j \in \mathbb{N}$. Therefore Corollary 2.2.4 together with the fact that $\tilde{\mathrm{e}}(\lambda|^{\delta_j})^{\wedge}(a) \neq 0$ for all $a \in E'$ (Property 2.4.10.1) yields the existence of a measure $\nu_j \in M^1(E)$ such that

$$\mu = \nu_j * \tilde{\mathrm{e}}(\lambda|^{\delta_j})$$

for all $j \in \mathbb{N}$. Now Theorem 2.2.21 implies that the sequence $(\mathrm{e}(\lambda|^{\delta_j}))_{j \geqslant 1}$ is τ_{w}-relatively compact, since the measures $\mathrm{e}(\lambda|^{\delta_j})$ are symmetric by Property 2.4.10.5. Finally, Theorem 3.3.7 yields the desired statement.

(iii) is shown in analogy to the proof of Theorem 3.3.4. $\qquad\square$

Theorem 3.3.13 (Construction of symmetric Lévy measures)
Let λ be a symmetric measure in $M^{\sigma}(E)$ satisfying the following conditions:

(a) $\lambda(\{0\}) = 0$.
(b) $\int (1 \wedge \|x\|) \, \lambda(\mathrm{d}x) < \infty$.

Then $\lambda \in \mathrm{L}_{\mathrm{s}}(E)$.

Proof. 1. Let $\lambda \in M^{\mathrm{b}}(E)$ such that $\int \|x\| \, \lambda(\mathrm{d}x) < \infty$. Then

$$\int \|x\| \, \mathrm{e}(\lambda)(\mathrm{d}x) \leqslant \int \|x\| \, \lambda(\mathrm{d}x) \,.$$

In order to see this we first establish the inequality

$$\int \|x\| \,\lambda^k(\mathrm{d}x) \leqslant k\lambda(E)^{k-1} \int \|x\| \,\lambda(\mathrm{d}x)$$

valid for all $k \in \mathbb{N}$. The proof runs by induction. While the case $k = 1$ is clear, we need only treat the step from k to $k+1$. But this is easily done:

$$
\begin{aligned}
\int \|x\| \,\lambda^{k+1}(\mathrm{d}x) &= \iint \|x+y\| \,\lambda^k(\mathrm{d}x)\,\lambda(\mathrm{d}y) \\
&\leqslant \iint \|x\| \,\lambda^k(\mathrm{d}x)\,\lambda(\mathrm{d}y) + \iint \|y\| \,\lambda^k(\mathrm{d}x)\,\lambda(\mathrm{d}y) \\
&= \lambda(E) \int \|x\| \,\lambda^k(\mathrm{d}x) + \lambda^k(E) \int \|y\| \,\lambda(\mathrm{d}y) \\
&\leqslant \lambda(E)k\lambda(E)^{k-1} \int \|x\| \,\lambda(\mathrm{d}x) + \lambda(E)^k \int \|x\| \,\lambda(\mathrm{d}x) \\
&= (k+1)\lambda(E)^k \int \|x\| \,\lambda(\mathrm{d}x)\,.
\end{aligned}
$$

Finally

$$
\begin{aligned}
\int \|x\| \,\mathrm{e}(\lambda)(\mathrm{d}x) &= \mathrm{e}^{-\lambda(E)} \sum_{k\geqslant 0} \frac{1}{k!} \int \|x\| \,\lambda^k(\mathrm{d}x) \\
&\leqslant \mathrm{e}^{-\lambda(E)} \sum_{k\geqslant 1} \frac{1}{k!} (k\lambda(E)^{k-1} \int \|x\| \,\lambda(\mathrm{d}x)) \\
&= \mathrm{e}^{-\lambda(E)} \left(\sum_{k\geqslant 1} \frac{1}{(k-1)!} \lambda(E)^{k-1} \right) \int \|x\| \,\lambda(\mathrm{d}x) \\
&= \int \|x\| \,\lambda(\mathrm{d}x)\,.
\end{aligned}
$$

Here we note that

$$\int \|x\| \,\lambda^0(\mathrm{d}x) = \int \|x\| \,\varepsilon_0(\mathrm{d}x) = 0\,.$$

2. From the assumptions on the symmetric measure λ we now deduce that $\lambda \in \mathrm{L_s}(E)$. We have $\lambda(\complement U_1) \leqslant \int (1 \wedge \|x\|)\,\lambda(\mathrm{d}x) < \infty$ and $\lambda = \lambda|_1 + \lambda|^1$. Without loss of generality we assume that $\lambda(\complement U_1) = 0$. With $\lambda_n := \lambda|^{1/n}$ for all $n \in \mathbb{N}$ we obtain that $\lambda_1 = 0$ and $\lambda_n \uparrow \lambda$. Moreover,

$$\lambda_n(E) = \lambda(\complement U_{1/n}) \leqslant n \int (1 \wedge \|x\|)\,\lambda(\mathrm{d}x) < \infty\,,$$

hence $\lambda \in M^\sigma(E)$ and $\lambda_n - \lambda_{n-1} \in M^{\mathrm{b}}(E)$ for all $n \geqslant 2$.

Let $(X_n)_{n \geqslant 2}$ be a sequence of independent E-valued random variables on a probability space $(\Omega, \mathfrak{A}, \mathsf{P})$ such that

$$\mathsf{P}_{X_n} := \mathrm{e}(\lambda_n - \lambda_{n-1}) \quad (n \geqslant 2).$$

Then

$$\mathsf{P}_{\sum_{j=m}^n X_j} = \underset{j=m}{\overset{n}{\LARGE *}} \mathsf{P}_{X_j}$$

$$= \underset{j=m}{\overset{n}{\LARGE *}} \mathrm{e}(\lambda_j - \lambda_{j-1})$$

$$= \mathrm{e}(\lambda_n - \lambda_{m-1}) \qquad (2 \leqslant m \leqslant n).$$

In particular

$$\mathsf{P}_{\sum_{j=2}^n X_j} = \mathrm{e}(\lambda_n) \qquad (n \geqslant 2).$$

But

$$\int \left\| \sum_{j=m}^n X_j \right\| \mathrm{d}\mathsf{P} = \int \|x\| \, \mathrm{e}(\lambda_n - \lambda_{m-1})(\mathrm{d}x)$$

$$\leqslant \int \|x\| \, (\lambda_n - \lambda_{m-1})(\mathrm{d}x) \qquad \text{(by part 1. of this proof)}$$

$$= \int_{\complement U_{1/n}} \|x\| \, \lambda(\mathrm{d}x) - \int_{\complement U_{1/(m-1)}} \|x\| \, \lambda(\mathrm{d}x)$$

$$\leqslant \int_{U_{1/(m-1)}} \|x\| \, \lambda(\mathrm{d}x).$$

Since $\int_{U_1} \|x\| \, \lambda(\mathrm{d}x) < \infty$, Lebesgue's dominated convergence theorem yields

$$\lim_{m \to \infty} \int_{U_{1/(m-1)}} \|x\| \, \lambda(\mathrm{d}x) = 0.$$

Now, the inequality

$$\mathsf{P}\left(\left[\left\| \sum_{j=m}^n X_j \right\| > \varepsilon \right] \right) \leqslant \frac{1}{\varepsilon} \int \left\| \sum_{j=m}^n X_j \right\| \mathrm{d}\mathsf{P}$$

valid for all $2 \leqslant m \leqslant n$ implies that

$$\lim_{m \to \infty} \sup_{n \geqslant m} \mathsf{P}\left(\left[\left\| \sum_{j=m}^n X_j \right\| > \varepsilon \right] \right) = 0$$

whatever $\varepsilon > 0$. Therefore

$$\sum_{j=2}^{m} X_j \xrightarrow{\text{P-stoch}} X\,,$$

where X is an E-valued random variable on $(\Omega, \mathfrak{A}, \mathsf{P})$, hence

$$\sum_{j=2}^{n} X_j \xrightarrow{\text{d}} \qquad (\text{as } n \to \infty)\,.$$

It follows that

$$\mathrm{e}(\lambda_n) \xrightarrow{\tau_{\mathrm{w}}}\,,$$

and from Theorem 3.3.7 that $\lambda \in \mathrm{L_s}(E)$ with $\mathsf{P}_X = \mathrm{e}(\lambda)$. $\qquad\square$

Discussion 3.3.14 of assumption (b) of Theorem 3.3.13.
There exist measures $\lambda \in \mathrm{L_s}(E)$ with

$$\int (1 \wedge \|x\|^2)\, \lambda(\mathrm{d}x) = \infty\,.$$

In fact, let $I := [0,1]$, $E := C(I)$ and $(x_n)_{n \geqslant 1}$ a sequence in $C(I)$ defined by $x_n(0) = x_n(2^{-n}) = x_n(2^{-(n-1)}) = x_n(1) = 0$, $x_n(\frac{3}{2}2^{-n}) = n^{-1/8}$ and extended linearly in between these arguments. Then $\|x_n\| = n^{-1/8}$ for all $n \in \mathbb{N}$. The measure

$$\lambda := \sum_{n \geqslant 1} n^{-3/4}(\varepsilon_{x_n} + \varepsilon_{-x_n}) \in M^\sigma(E)$$

is symmetric and satisfies $\lambda(\{0\}) = 0$. Moreover,

$$\int_{U_1} \|x\|^2\, \lambda(\mathrm{d}x) = \sum_{n \geqslant 1} n^{-3/4} 2 n^{-1/4} = 2\sum_{n \geqslant 1} n^{-1} = \infty$$

and $\lambda(\complement U_\delta) < \infty$ for all $\delta \in \mathbb{R}_+^\times$.

We now consider sequences $(\xi_n)_{n \geqslant 1}$ and $(\xi'_n)_{n \geqslant 1}$ of independent real-valued random variables on $(\Omega, \mathfrak{A}, \mathsf{P})$ with

$$\mathsf{P}_{\xi_n} = \mathsf{P}_{\xi'_n} = \Pi(n^{-3/4})$$

for all $n \in \mathbb{N}$. Then the E-valued random variables $\xi_n x_n$ and $\xi'_n x_n$ are independent and have distributions

$$\mathsf{P}_{\xi_n x_n} = \mathsf{P}_{\xi'_n x_n} = \mathrm{e}(n^{-3/4}\varepsilon_{x_n}) \quad (n \in \mathbb{N})\,.$$

In fact, if ξ is a real-valued random variable on $(\Omega, \mathfrak{A}, \mathsf{P})$ with $\mathsf{P}_\xi = \Pi(\alpha)$ $(= \mathrm{e}(\alpha\varepsilon_1))$ and $y \in E$ then

$$\mathsf{P}_{\xi y} = \mathrm{e}(\alpha\varepsilon_y)\,,$$

since for all $a \in E'$ we have

$$
\begin{aligned}
(\mathsf{P}_{\xi y})^{\wedge}(a) &= \int e^{i\langle x, a \rangle} \, \mathsf{P}_{\xi y}(dx) = \int e^{i\xi(\omega)\langle y, a \rangle} \, \mathsf{P}(d\omega) \\
&= \int e^{it\langle y, a \rangle} \, \mathsf{P}_{\xi}(dt) = \int e^{it\langle y, a \rangle} \, \Pi(\alpha)(dt) \\
&= \Pi(\alpha)^{\wedge}(\langle y, a \rangle) = \exp(\alpha(e^{i\langle y, a \rangle} - 1)) = e(\alpha \varepsilon_y)^{\wedge}(a) \, .
\end{aligned}
$$

Next we obtain that

$$
\begin{aligned}
\mathsf{P}_{Y_n} := \mathsf{P}_{\xi_n x_n - \xi'_n x_n} \\
&= e\big(n^{-3/4} \varepsilon_{x_n}\big) * e\big(n^{-3/4} \varepsilon_{x_n}\big)^{-} \\
&= e\big(n^{-3/4}(\varepsilon_{x_n} + \varepsilon_{-x_n})\big) \qquad (n \in \mathbb{N}) \, ,
\end{aligned}
$$

where the $C(I)$-valued random variables $Y_n := \xi_n x_n - \xi'_n x_n$ are independent (on $(\Omega, \mathfrak{A}, \mathsf{P})$). It follows that

$$
\begin{aligned}
\mathsf{P}_{\sum_{j=1}^{n} Y_j} &= \mathop{\Large *}_{j=1}^{n} \mathsf{P}_{Y_j} = \mathop{\Large *}_{j=1}^{n} e\big(j^{-3/4}(\varepsilon_{x_j} + \varepsilon_{-x_j})\big) \\
&= e\left(\sum_{j=1}^{n} j^{-3/4}(\varepsilon_{x_j} + \varepsilon_{-x_j}) \right) = e\left(\lambda|^{(n+1)^{-1/8}} \right) = e\left(\lambda|^{\delta_n} \right) \, ,
\end{aligned}
$$

where $\delta_n := (n+1)^{-1/8}$ for all $n \in \mathbb{N}$. But now

$$
[|\xi_n - \xi'_n| > 1] \subset [\xi_n > 1] \cup [\xi'_n > 1] \, ,
$$

hence

$$
\begin{aligned}
\sum_{n \geqslant 1} \mathsf{P}([|\xi_n - \xi'_n| > 1]) &\leqslant 2 \sum_{n \geqslant 1} \mathsf{P}([\xi_n > 1]) \\
&= 2 \sum_{n \geqslant 1} \Pi(n^{-3/4}(\{2, 3, \dots\})) \leqslant 2 \sum_{n \geqslant 1} n^{-3/2} < \infty \, .
\end{aligned}
$$

Here we use the estimate $\Pi(\alpha)(\{2, 3, \dots\}) \leqslant \alpha^2$ which is obviously obtained from

$$
\Pi(\alpha)(\{0, 1\}) = e^{-\alpha}(1 + \alpha) \geqslant (1 - \alpha)(1 + \alpha) = 1 - \alpha^2
$$

for $\alpha \in \mathbb{R}_+^{\times}$. Now the Borel–Cantelli Lemma applies and yields

$$
\mathsf{P}(\limsup_{n \to \infty}[|\xi_n - \xi'_n| > 1]) = 0
$$

or

$$
\mathsf{P}(\liminf_{n \to \infty}[|\xi_n - \xi'_n| \leqslant 1]) = 1 \, .
$$

Therefore, for P-almost all $\omega \in \Omega$ there exists an $n(\omega) \in \mathbb{N}$ such that $|\xi_n(\omega) - \xi_n'(\omega)| \leqslant 1$ whenever $n \geqslant n(\omega)$. This implies that for $n \geqslant m \geqslant n(\omega)$

$$\left\| \sum_{j=m}^{n} Y_j(\omega) \right\| = \left\| \sum_{j=m}^{n} (\xi_j(\omega) - \xi_j'(\omega)) x_j \right\| \leqslant m^{-1/8} ,$$

since $[x_j \neq 0] \cap [x_k \neq 0] = \emptyset$ for $j \neq k$. But then

$$\sum_{j \geqslant 1} Y_j \xrightarrow{\text{P-a.s.}} Y ,$$

where Y is a $C(I)$-valued random variable on $(\Omega, \mathfrak{A}, \mathsf{P})$, hence

$$e(\lambda|^{\delta_n}) = \mathsf{P}_{\sum_{j=1}^{n} Y_j} \xrightarrow{\tau_{\mathrm{w}}} \mathsf{P}_Y .$$

Theorem 3.3.7 yields that $\lambda \in \mathrm{L_s}(E)$ with $e(\lambda) = \mathsf{P}_Y$.

3.4   The Lévy–Khinchin decomposition

The aim of the subsequent discussion is to establish the canonical decomposition of infinitely divisible probability measures on a separable Banach space as convolutions of measures of Poisson type, Gauss measures and of Dirac measures. The first named measures will be defined via not necessarily symmetric Lévy measures. For symmetric Lévy measures the canonical decomposition can be derived from the material of the previous section. In the nonsymmetric case the proof depends on an additional technique: the centralization of generalized Poisson measures.

For any measure $\lambda \in M^{\mathrm{b}}(E)$ we consider the Bochner integral $x(\lambda) \in E$ given by

$$x(\lambda) := -\int_{U_1} x \, \lambda(\mathrm{d}x)$$

Theorem 3.4.1 *Let* $(\lambda_n)_{n \geqslant 1}$ *be a sequence in* $M^{\mathrm{b}}(E)$ *such that the sequence* $(\mathrm{e}(\lambda_n))_{n \geqslant 1}$ *is relatively shift compact. Then the sequence* $(\mathrm{e}(\lambda_n) * \varepsilon_{x(\lambda_n)})_{n \geqslant 1}$ *is* τ_{w}-*relatively compact.*

Proof.   Applying Theorem 3.3.3 it suffices to show that the sequence $(\mathrm{e}(\lambda_n|^1) * \varepsilon_{x(\lambda_n)})_{n \geqslant 1}$ is τ_{w}-relatively compact. We may therefore assume without loss of generality that $\lambda_n(\complement U_1) = 0$ for all $n \geqslant 1$. From the inequalities

$$\left| 1 - (\mathrm{e}(\lambda_n) * \varepsilon_{x(\lambda_n)})^{\wedge}(a) \right| = \left| 1 - \exp\left(\int_{U_1} (\mathrm{e}^{\mathrm{i}\langle x,a\rangle} - 1 - \mathrm{i}\langle x,a\rangle)\, \lambda_n(\mathrm{d}x) \right) \right|$$

$$\leqslant \exp\left(\frac{1}{2} \int_{U_1} \langle x,a\rangle^2 \, \lambda_n(\mathrm{d}x) \right) - 1$$

valid for all $a \in V_1$ we deduce that the sequence $\left((\mathrm{e}(\lambda_n) * \varepsilon_{x(\lambda_n)})^{\wedge} \right)_{n \geqslant 1}$ is equicontinuous in $C(V_1)$. Theorem 3.3.4 together with Theorem 2.2.12 yield the assertion.   □

Definition 3.4.2 For any (bounded) measure $\lambda \in M^{\mathrm{b}}(E)$ the measure

$$\mathrm{e}_{\mathrm{s}}(\lambda) := \mathrm{e}(\lambda) * \varepsilon_{x(\lambda)}$$

is called the **exponential** of λ.

Introducing the kernel K by

$$K(x,a) := \mathrm{e}^{\mathrm{i}\langle x,a\rangle} - 1 - \mathrm{i}\langle x,a\rangle \mathbb{1}_{U_1}(x)$$

for all $x \in E,\ a \in E'$ we observe that

$$\mathrm{e}_{\mathrm{s}}(\lambda)^{\wedge}(a) = \exp\left(\int_E K(x,a)\, \lambda(\mathrm{d}x) \right)$$

whenever $a \in E'$.

Clearly, $\mathrm{e_s}(\lambda) \in \mathrm{I}(E)$ and $\mathrm{e_s}(\lambda) = \mathrm{e}(\lambda)$ provided λ is symmetric. In this case $x(\lambda) = 0$.

Comparing the Fourier transform of the exponential $\mathrm{e_s}(\lambda)$ of λ with that of the Poisson measure $\mathrm{e}(\lambda)$ with exponent λ the desired generalization to nonsymmetric measures λ relies on a systematic replacement of the kernel $(x, a) \mapsto \cos\langle x, a \rangle - 1$ by the kernel K.

Theorem 3.4.3 *Let* $(\lambda_n)_{n \geqslant 1}$ *be a* τ_{w}-*relatively compact sequence in* $M^{\mathrm{b}}(E)$. *Then for any* $\delta > 0$ *the sequence* $(x_n)_{n \geqslant 1}$ *defined by*

$$x_n = x_n^{\delta} := \int_{U_\delta} x \, \lambda_n(\mathrm{d}x)$$

for all $n \in \mathbb{N}$ *is relatively compact in* E.

Proof. From Prohorov's theorem 1.3.7 we infer that for $\varepsilon > 0$ there exists a compact, convex and balanced set $K \subset E$ such that

$$\lambda_n(\complement K) < \frac{\varepsilon}{\delta}$$

for all $n \in \mathbb{N}$. But this implies that

$$\left\| \int_{U_\delta \cap \complement K} x \, \lambda_n(\mathrm{d}x) \right\| < \varepsilon$$

for all $n \in \mathbb{N}$. Thus it remains to show that the sequence $(y_n)_{n \geqslant 1}$ defined by

$$y_n := \int_{U_\delta \cap K} x \, \lambda_n(\mathrm{d}x)$$

for all $n \in \mathbb{N}$ admits a finite ε-net. In fact, if this statement has been established, the sequence $(x_n)_{n \geqslant 1}$ admits a finite 2ε-net, and this being true for every $\varepsilon > 0$ yields the assertion.

In order to finish the proof we just consider an $a \in E'$ such that $|\langle x, a \rangle| \leqslant 1$ for all $x \in K$. We obtain that

$$|\langle y_n, a \rangle| \leqslant \lambda_n(K) \leqslant \lambda_n(E) \leqslant d := \sup_{n \geqslant 1} \lambda_n(E),$$

hence by an application of the bipolar theorem that $y_n \in dK$ for all $n \in \mathbb{N}$. But this implies that $(y_n)_{y \geqslant 1}$ is relatively compact. $\qquad\square$

Corollary 3.4.4 *Let* $(\lambda_n)_{n \geqslant 1}$ *be a sequence in* $M^{\mathrm{b}}(E)$ *such that*

$$\lambda_n \xrightarrow{\tau_{\mathrm{w}}} \lambda.$$

Then for every $\delta \in C(\lambda)$ *we have*

$$x_n := \int_{U_\delta} x\,\lambda_n(\mathrm{d}x) \to \int_{U_\delta} x\,\lambda(\mathrm{d}x)$$

as $n \to \infty$.

Proof. From the theorem we know that $(x_n)_{n \geqslant 1}$ being relatively compact admits a subsequence $(x_{n'})$ such that

$$\langle x_{n'}, a \rangle \to \langle y, a \rangle$$

for $y \in E$ whenever $a \in E'$. On the other hand Theorem 1.2.9 implies that

$$\langle x_{n'}, a \rangle \to \int_{U_\delta} \langle x, a \rangle\,\lambda(\mathrm{d}x)$$

for all $a \in E'$. Since the limits of $(\langle x_{n'}, a \rangle)$ coincide for all subsequences of $(x_n)_{n \geqslant 1}$, the assertion has been proved. □

Facts 3.4.5 of exponentials of bounded measures.

3.4.5.1 *For* $(\lambda_n)_{n \geqslant 1}$ *in* $M^{\mathrm{b}}(E)$ *the sequence* $(\mathrm{e}_{\mathrm{s}}(\lambda_n))_{n \geqslant 1}$ *is relatively shift compact if and only if it is* τ_{w}*-relatively compact, and both properties are equivalent to the relative shift compactness of the sequence* $(\mathrm{e}(\lambda_n))_{n \geqslant 1}$.

This fact follows directly from Theorem 3.4.1.

3.4.5.2 *For* $(\lambda_n)_{n \geqslant 1}$ *in* $M^{\mathrm{b}}(E)$ *with* $\lambda_n \xrightarrow{\tau_{\mathrm{w}}} \lambda$ *and* $1 \in C(\lambda)$ *we have*

$$\mathrm{e}_{\mathrm{s}}(\lambda_n) \xrightarrow{\tau_{\mathrm{w}}} \mathrm{e}_{\mathrm{s}}(\lambda).$$

This statement is a restricted version of the τ_{w}-continuity of the mapping $\lambda \mapsto \mathrm{e}_{\mathrm{s}}(\lambda)$ from $M^{\mathrm{b}}(E)$ into $M^1(G)$ the restriction being crucial.

As for the proof of Fact 3.4.5.2 we just observe that

$$\mathrm{e}_{\mathrm{s}}(\lambda_n) = \mathrm{e}(\lambda_n) * \varepsilon_{x(\lambda_n)},$$

where

$$x(\lambda_n) := -\int_{U_1} x\,\lambda_n(\mathrm{d}x)$$

for all $n \geqslant 1$, and apply Theorem 2.4.12 (ii) together with Corollary 3.4.4.

Theorem 3.4.6 *For any sequence* $(\lambda_n)_{n \geqslant 1}$ *in* $M^{\mathrm{b}}(E)$ *with* $\lambda_n \xrightarrow{\tau_{\mathrm{w}}} \lambda$ *the following statements are equivalent:*

(i) $e_s(\lambda_n) \xrightarrow{\tau_w}$ $(in\ M^1(E))$.
(ii) $x(\lambda_n) \to$ $(in\ E)$ $(as\ n \to \infty)$.

Moreover, if either of these statements is available, we have that

$$e_s(\lambda_n) \xrightarrow{\tau_w} e_s(\lambda) * \varepsilon_z \,,$$

where

$$z = \lim_{n\to\infty} x(\lambda_n) - x(\lambda)$$

$$= \int_{U_1} x\,\lambda(\mathrm{d}x) - \lim_{n\to\infty} \int_{U_1} x\,\lambda_n(\mathrm{d}x)$$

$$= \lim_{\substack{\delta\downarrow 1 \\ \delta\in C(\lambda)}} \lim_{n\to\infty} \int_{U_\delta \cap \complement U_1} x\,\lambda_n(\mathrm{d}x)\,.$$

Clearly, if $1 \in C(\lambda)$ *then* $z = 0$.

Proof. For every $n \geqslant 1$ we have that

$$e_s(\lambda_n) = e(\lambda_n) * \varepsilon_{x(\lambda_n)}\,.$$

Now we apply Theorem 2.4.12 (ii) together with Corollary 2.2.4 in order to obtain the equivalence (i) $\Longleftrightarrow$ (ii). As to the remaining statements of the theorem we suppose that

$$e_s(\lambda_n) \xrightarrow{\tau_w} \mu$$

for some $\mu \in M^1(E)$. But then $\lim_{n\to\infty} x(\lambda_n)$ exists, and

$$\mu = \lim_{n\to\infty}\left(e(\lambda_n) * \varepsilon_{x(\lambda_n)}\right)$$

$$= e(\lambda) * \varepsilon_{\lim_{n\to\infty} x(\lambda_n)}$$

$$= e_s(\lambda) * \varepsilon_{\lim_{n\to\infty} x(\lambda_n) - x(\lambda)}\,.$$

The double limit representation of $z = \lim_{n\to\infty} x(\lambda_n) - x(\lambda)$ follows from Corollary 3.4.4. □

Definition 3.4.7 A measure $\lambda \in M^\sigma(E)$ is called a **Lévy measure** if $\lambda + \lambda^-$ is a symmetric Lévy measure (in the sense of Definition 3.3.1).

The totality of Lévy measures will be abbreviated by $L(E)$.

The following

Properties 3.4.8 of a Lévy measure λ are proved similarly to those for a symmetric one (see Properties 3.3.6).

3.4.8.1 $\lambda(\{0\}) = 0$.

3.4.8.2 *For each $\delta > 0$ we have that $\lambda(\complement U_\delta) < \infty$.*

3.4.8.3 *For $(\delta_n)_{n \geqslant 1}$ in $\mathbb{R}_+^\times$ with $\delta_n \downarrow 0$ the sequence $(\mathrm{e}(\lambda|^{\delta_n}))_{n \geqslant 1}$ is relatively shift compact.*

3.4.8.4 $\sup \left\{ \int_{U_1} \langle x, a \rangle^2 \, \lambda(\mathrm{d}x) : a \in V_1 \right\} < \infty$.

3.4.8.5 *For $(\delta_n)_{n \geqslant 1}$ in $\mathbb{R}_+^\times$ with $\delta_n \downarrow 0$ the sequence $(\mathrm{e}(\lambda|^{\delta_n}))_{n \geqslant 1}$ is τ_{w}-relatively compact.*

3.4.8.6 *If $\sigma \in M^\sigma(E)$ satisfies $\sigma \leqslant \lambda$, then σ is also a Lévy measure.*

The only properties deserving additional arguments are 3.4.8.3 and 3.4.8.5. For 3.4.8.3 we note that by Theorem 3.3.7 the sequence $(\mathrm{e}(\lambda + \lambda^-)|^{\delta_n})_{n \geqslant 1}$ is τ_{w}-relatively compact. But since

$$\mathrm{e}(\lambda|^{\delta_n}) \prec \mathrm{e}((\lambda + \lambda^-)|^{\delta_n})$$

for all $n \geqslant 1$, the sequence $(\mathrm{e}(\lambda|^{\delta_n}))_{n \geqslant 1}$ is relatively shift compact. Property 3.4.8.5 is a direct consequence of Fact 3.4.5.1.

Theorem 3.3.7 can be extended to nonsymmetric Lévy measures.

Theorem 3.4.9 (Characterization of Lévy measures)
For any $\lambda \in M^\sigma(E)$ satisfying $\lambda(\{0\}) = 0$ the following statements are equivalent:

(i) $\lambda \in \mathrm{L}(E)$.

(ii) (a) *For each $a \in E'$*

$$\int |K(x, a)| \, \lambda(\mathrm{d}x) < \infty \,,$$

    and

    (b) *there exists a measure $\tilde{\mathrm{e}}_{\mathrm{s}}(\lambda) \in M^1(E)$ such that*

$$\widehat{\tilde{\mathrm{e}}_{\mathrm{s}}(\lambda)}(a) = \exp\left(\int K(x, a) \, \lambda(\mathrm{d}x) \right)$$

    for all $a \in E'$.

(iii) *There exists a sequence $(\lambda_n)_{n \geqslant 1}$ in $M^{\mathrm{b}}(E)$ with $\lambda_n \uparrow \lambda$ such that $(\mathrm{e}_{\mathrm{s}}(\lambda_n))_{n \geqslant 1}$ is τ_{w}-relatively compact.*

(iv) *For each $\delta > 0$*

    (a) $\lambda(\complement U_\delta) < \infty$,

    and

(b) *for some (each) sequence* $(\delta_n)_{n \geqslant 1}$ *with* $\delta_n \downarrow 0$ *the sequence* $(\mathrm{e_s}(\lambda|^{\delta_n}))_{n \geqslant 1}$ *is* τ_w-*relatively compact.*

Proof.   As the implication (i) $\Rightarrow$ (iv) follows with the help of Properties 3.4.8, (iv) $\Rightarrow$ (iii) is trivial, and (iii) $\Rightarrow$ (ii) is deduced from Properties 3.3.6 (as in the proof of Theorem 3.3.7) by adapting the arguments to the measures $\mathrm{e_s}(\lambda)$ and the kernel K, it remains to show implication (ii) $\Rightarrow$ (i). For completeness we prove the full equivalence (i) $\Leftrightarrow$ (ii).

(i) $\Rightarrow$ (ii).   Let $\lambda \in \mathrm{L}(E)$. From Properties 3.4.8.2 and 3.4.8.4 we conclude that

$$\int |K(x,a)|\, \lambda(\mathrm{d}x) < \infty$$

for all $a \in E'$, since $|K(x,a)| \leqslant \frac{1}{2}\langle x,a\rangle^2$ for all $x \in E$ satisfying $\|x\| \leqslant 1$. Moreover, choosing a sequence $(\delta_n)_{n \geqslant 1}$ in $\mathbb{R}_+^\times$ with $\delta_n \downarrow 0$ and putting $\lambda_n := \lambda|^{\delta_n}$ for all $n \geqslant 1$ we obtain that

$$\lim_{n \to \infty} \int K(x,a)\, \lambda_n(\mathrm{d}x) = \int K(x,a)\, \lambda(\mathrm{d}x)$$

for each $a \in E'$. Now Property 3.4.8.5 can be employed in order to establish the τ_w-convergence of the sequence $(\mathrm{e_s}(\lambda_n))_{n \geqslant 1}$ towards a measure $\tilde{\mathrm{e}}_\mathrm{s}(\lambda) \in M^1(E)$ such that

$$\widehat{\tilde{\mathrm{e}}_\mathrm{s}(\lambda)}(a) = \exp\Big(\int K(x,a)\, \lambda(\mathrm{d}x) \Big)$$

for all $a \in E'$.

(ii) $\Rightarrow$ (i).   Let $\lambda \in M^\sigma(E)$ with $\lambda(\{0\}) = 0$ satisfy the conditions (a) and (b) of (ii). Along with λ also λ^- satisfies (ii), hence the function

$$a \mapsto \exp\Big(\int (\cos\langle x,a\rangle - 1)\,(\lambda + \lambda^-)(\mathrm{d}x) \Big)$$

on E' is the Fourier transform of the measure $\tilde{\mathrm{e}}_\mathrm{s}(\lambda) * \tilde{\mathrm{e}}_\mathrm{s}(\lambda^-) \in M^1(E)$. This proves (i). $\qquad\square$

Definition 3.4.10 The measure $\tilde{\mathrm{e}}_\mathrm{s}(\lambda) \in M^1(E)$ introduced in (ii) of Theorem 3.4.9 is called the **generalized exponential of the Lévy measure** λ.

Clearly, $\tilde{\mathrm{e}}_\mathrm{s}(\lambda) = \tilde{\mathrm{e}}(\lambda)$ whenever $\lambda \in \mathrm{L_s}(E)$.

Properties 3.4.11   of the generalized exponential mapping.

3.4.11.1   *The mapping* $\tilde{\mathrm{e}}_\mathrm{s}$ *from* $\mathrm{L}(E)$ *into* $M^1(E)$ *is an involutive semigroup homomorphism, i.e.*

(a) $L(E) + L(E) \subset L(E)$, *and*

$$\tilde{e}_s(\lambda_1 + \lambda_2) = \tilde{e}_s(\lambda_1) * \tilde{e}_s(\lambda_2)$$

whenever $\lambda_1, \lambda_2 \in L(E)$.

(b) $L(E)^- \subset L(E)$, *and*

$$\tilde{e}_s(\lambda^-) = \tilde{e}_s(\lambda)^-$$

whenever $\lambda \in L(E)$.

3.4.11.2 $\tilde{e}_s(L(E)) \subset EM(E)$, *i.e. for every* $\lambda \in L(E)$ *the generalized exponential* $\tilde{e}_s(\lambda)$ *of* λ *is (continuously) embeddable with (continuous) embedding semigroup* $(\tilde{e}_s(\lambda)_t)_{t \geqslant 1}$ *given by*

$$\tilde{e}_s(\lambda)_t := \tilde{e}_s(t\lambda)$$

for all $t \in \mathbb{R}$.

3.4.11.3 *For any* $\lambda_1, \lambda_2 \in L(E)$ *with* $\lambda_1 \leqslant \lambda_2$ *we have*

$$\tilde{e}_s(\lambda_1) \prec \tilde{e}_s(\lambda_2) \,.$$

3.4.11.4 *Let* $(\lambda_n)_{n \geqslant 1}$ *be a sequence in* $M^b(E)$ *with* $\lambda_n \uparrow \lambda$. *Then*

$$\tilde{e}_s(\lambda_n) = e_s(\lambda_n) \xrightarrow{\tau_w} \tilde{e}_s(\lambda) \,.$$

3.4.11.5 *For any sequence* $(\lambda_n)_{n \geqslant 1}$ *in* $L(E)$ *the sequence* $(\tilde{e}_s(\lambda_n))_{n \geqslant 1}$ *is* τ_w-*relatively compact provided it is relatively shift compact.*

3.4.11.6 *For any sequence* $(\lambda_n)_{n \geqslant 1}$ *in* $L(E)$ *such that* $(\tilde{e}_s(\lambda_n))_{n \geqslant 1}$ *is relatively shift compact the sequence* $(\lambda_n|^\delta)_{n \geqslant 1}$ *is* τ_w-*relatively compact whenever* $\delta > 0$.

While Properties 3.4.11.1 to 3.4.11.4 are obvious (in particular with respect to their analogs in the symmetric case), we need only note that Property 3.4.11.6 follows from Theorem 3.3.3 together with Property 2.2.16.5, and that Property 3.4.11.5 is a consequence of Theorem 3.3.4 in whose proof the boundedness of the Lévy measures has not been used.

After all these preparations we are ready to approach the Lévy–Khinchin decomposition. We start with the uniqueness of the decomposition whose proof relies on a modification of Lemma 3.3.10 and Theorem 3.3.11.

Given $\lambda \in L(E)$ and $a \in E'$ we introduce the measure λ^a by

$$\lambda^a(B) := \int_B (1 - \cos\langle x, a \rangle) \, \lambda(dx)$$

for all $B \in \mathfrak{B}(E)$. From Properties 3.4.8.2 and 3.4.8.4 we infer that $\lambda^a \in M^b(E)$.

Lemma 3.4.12 *Let* $\lambda_1, \lambda_2 \in L(E)$ *satisfying* $\lambda_1^a = \lambda_2^a$ *for all* $a \in E'$. *Then* $\lambda_1 = \lambda_2$.

Proof.   The assumption yields the equality

$$(\lambda_1|^\delta)^a = (\lambda_2|^\delta)^a$$

for each $\delta > 0$ which, once

$$\lambda_1|^\delta = \lambda_2|^\delta$$

has been shown, implies that $\lambda_1 = \lambda_2$. It suffices therefore to perform the proof of the assertion for $\lambda_1, \lambda_2 \in \mathrm{L}(E) \cap M^{\mathrm{b}}(E)$. But in this case the proof of Lemma 3.3.10 takes care of the remaining reasoning.   □

Theorem 3.4.13 *The generalized exponential mapping $\tilde{\mathrm{e}}_{\mathrm{s}} : \mathrm{L}(E) \to M^1(E)$ is injective. Moreover, let $\lambda_1, \lambda_2 \in \mathrm{L}(E)$ and $x_0 \in E$ such that*

$$\tilde{\mathrm{e}}_{\mathrm{s}}(\lambda_1) = \tilde{\mathrm{e}}_{\mathrm{s}}(\lambda_2) * \varepsilon_{x_0} \, .$$

Then $\lambda_1 = \lambda_2$ and $x_0 = 0$.

Proof.   We modify the arguments applied in the proof of Theorem 3.3.11 in the obvious manner. From the assumption we have that

$$\widehat{\tilde{\mathrm{e}}_{\mathrm{s}}(\lambda_1)}(a) = \widehat{\tilde{\mathrm{e}}_{\mathrm{s}}(\lambda_2)}(a)\mathrm{e}^{\mathrm{i}\langle x_0, a\rangle}$$

for all $a \in E'$. On the other hand

$$K(x, a+b) + K(x, a-b) - 2K(x, a) = 2\mathrm{e}^{\mathrm{i}\langle x, a\rangle}(\cos\langle x, b\rangle - 1)$$

whenever $x \in E$, $a, b \in E'$. Consequently

$$\exp\left(2\int \mathrm{e}^{\mathrm{i}\langle x, a\rangle}(\cos\langle x, b\rangle - 1)\,\lambda_1(\mathrm{d}x)\right)$$
$$= \exp\left(2\int \mathrm{e}^{\mathrm{i}\langle x, a\rangle}(\cos\langle x, b\rangle - 1)\,\lambda_2(\mathrm{d}x)\right),$$

hence

$$\widehat{\lambda_1^b}(a) - \widehat{\lambda_2^b}(a) = \mathrm{i}\pi k(a, b)$$

with $k(a, b) \in \mathbb{Z}$ for $a, b \in E'$. But since $k(a, b)$ turns out to be 0 for all $b \in E'$ we obtain $\lambda_1^b = \lambda_2^b$ for all $b \in E'$, which by Lemma 3.4.12 implies that $\lambda_1 = \lambda_2$ and clearly $x_0 = 0$.   □

Lemma 3.4.14 *Let $\lambda \in \mathrm{L}(E)$.   Then*

$$\lim_{t \to 0} t^2 \int K\left(x, \frac{a}{t}\right) \lambda(\mathrm{d}x) = 0$$

whenever $a \in E'$.

Proof.   For each $x \in \complement U_1$ we have $|K(x,a)| \leqslant 2$, hence

$$\lim_{t \to 0} t^2 \int_{\complement U_1} \left| K\left(x, \frac{a}{t}\right) \right| \lambda(\mathrm{d}x) \leqslant 2 \lim_{t \to 0} t^2 \, \lambda(\complement U_1) = 0 \qquad (a \in E').$$

As a consequence of this it suffices to study the integral

$$\int_{U_1} K\left(x, \frac{a}{t}\right) \lambda(\mathrm{d}x)$$

for $a \in E'$. If $x \in U_1$ we have

$$\left| K\left(x, \frac{a}{t}\right) \right| \leqslant \frac{\langle x, a \rangle^2}{2t^2}$$

and therefore

$$\left| t^2 K\left(x, \frac{a}{t}\right) \right| \leqslant \frac{\langle x, a \rangle^2}{2}$$

for all $x \in E$, $a \in E'$ and $t > 0$. On the other hand

$$\lim_{t \to 0} t^2 K\left(x, \frac{a}{t}\right) = 0$$

for all $x \in E$ and $a \in E'$. Property 3.4.8.4 enables us to apply the dominated convergence theorem, and the assertion has been proved. $\square$

Theorem 3.4.15 (Uniqueness of the canonical decomposition)
Let $\varrho_1, \varrho_2 \in \mathrm{G}(E)$, $\lambda_1, \lambda_2 \in \mathrm{L}(E)$ *and* $x_1, x_2 \in E$ *such that*

$$\varrho_1 * \tilde{\mathrm{e}}_{\mathrm{s}}(\lambda_1) * \varepsilon_{x_1} = \varrho_2 * \tilde{\mathrm{e}}_{\mathrm{s}}(\lambda_2) * \varepsilon_{x_2}.$$

Then $\varrho_1 = \varrho_2$, $\lambda_1 = \lambda_2$ *and* $x_1 = x_2$.

Proof.   By Theorem 2.4.7 there exists for $j = 1, 2$ a mapping $q_j : E' \to \mathbb{R}_+$ with the property

$$q_j(ta) = t^2 q_j(a)$$

whenever $t \in \mathbb{R}$ such that $\hat{\varrho}_j(a) = \exp(-q_j(a))$ for all $a \in E'$. From the assumption we deduce the equality

$$\hat{\varrho}_1(a)\, \tilde{\mathrm{e}}_{\mathrm{s}}(\lambda_1)^{\wedge}(a)\, \hat{\varepsilon}_{x_1}(a) = \hat{\varrho}_2(a)\, \tilde{\mathrm{e}}_{\mathrm{s}}(\lambda_2)^{\wedge}(a)\, \hat{\varepsilon}_{x_2}(a)$$

and by taking logarithms the equality

$$-q_1(a) + \int K(x,a)\, \lambda_1(\mathrm{d}x) + \mathrm{i}\langle x_1, a \rangle$$

$$= -q_2(a) + \int K(x,a)\, \lambda_2(\mathrm{d}x) + \mathrm{i}\langle x_2, a \rangle$$

valid for all $a \in E'$. For given $t > 0$ we now replace $a \in E'$ by $\frac{a}{t}$ and multiply both sides of the above equality by t^2. Taking limits for $t \to 0$ the limit relationship of Lemma 3.4.14 implies $q_1(a) = q_2(a)$, hence $\hat{\varrho}_1(a) = \hat{\varrho}_2(a)$ for all $a \in E'$ and therefore by the uniqueness of the Fourier transform that $\varrho_1 = \varrho_2$. Finally Theorem 3.4.13 implies that $\lambda_1 = \lambda_2$ and $x_1 = x_2$. $\square$

Remark 3.4.16 If $\tilde{e}_s(\lambda) * \varepsilon_{x_0} \in G(E)$ for some $\lambda \in L(E)$ and $x_0 \in E$, then $\lambda = 0$ and $x_0 = 0$.

Theorem 3.4.17 Let $(\lambda_n)_{n \geqslant 1}$ be a sequence in $L(E)$ with

$$\tilde{e}_s(\lambda_n) \xrightarrow{\tau_w} \mu \in M^1(E).$$

Moreover, for every $\delta \in \mathbb{R}_+^\times$ let there exist an $n(\delta) \in \mathbb{N}$ such that $\lambda_n(\complement U_\delta) = 0$ for all $n \geqslant n(\delta)$. Then

(i) $\mu \in G(E)$.
(ii) $\hat{\mu}(a) = \exp(-\frac{1}{2} \lim_{n \to \infty} \int \langle x, a \rangle^2 \, \lambda_n(dx))$ whenever $a \in E'$.

Proof. From Theorem 3.3.12 (iii) (obviously valid also for nonsymmetric Lévy measures) we infer that

$$\alpha := \sup\left\{ \int_{U_1} \langle x, a \rangle^2 \, \lambda_n(dx) : a \in V_1, \ n \in \mathbb{N} \right\} < \infty.$$

For all $a \in E'$ and $n \geqslant n(\delta)$ we therefore obtain that

$$\int_E |\langle x, a \rangle|^3 \, \lambda_n(dx) = \int_{U_\delta} |\langle x, a \rangle|^3 \, \lambda_n(dx) \leqslant \alpha \delta \|a\|^3.$$

This shows that

$$\lim_{n \to \infty} \int_E |\langle x, a \rangle|^3 \, \lambda_n(dx) = 0 \qquad (1)$$

whenever $a \in E'$. Along with K we now consider the kernel M defined by

$$M(x, a) := K(x, a) + \frac{1}{2}\langle x, a \rangle^2$$

for all $x \in E$, $a \in E'$. It follows from the Taylor expansion of the exponential function that (1) implies

$$\lim_{n \to \infty} \int_E M(x, a) \, \lambda_n(dx) = 0$$

for all $a \in E'$. Now we employ Theorems 2.3.7, 2.3.3 and Corollary 2.1.16 in order to obtain that

$$\begin{aligned}
\mathrm{Log}\, \hat{\mu}(a) &= \lim_{n \to \infty} \int K(x, a) \, \lambda_n(dx) \\
&= \lim_{n \to \infty} \left(\int M(x, a) \, \lambda_n(dx) - \frac{1}{2} \int \langle x, a \rangle^2 \, \lambda_n(dx) \right) \\
&= \lim_{n \to \infty} \left(-\frac{1}{2} \int \langle x, a \rangle^2 \, \lambda_n(dx) \right)
\end{aligned}$$

whenever $a \in E'$. The mapping $q : E' \to \mathbb{R}$ defined by

$$q(a) := \frac{1}{2} \lim_{n \to \infty} \int \langle x, a \rangle^2 \, \lambda_n(\mathrm{d}x)$$

for all $a \in E'$ obviously has the property that

$$q(ta) = t^2 q(a)$$

valid for all $a \in E'$, $t \in \mathbb{R}$. Consequently by Theorem 2.4.7 $\mu \in \mathrm{G}(E)$.

$\square$

Theorem 3.4.18 *Let* $(\lambda_n)_{n \geqslant 1}$ *be a sequence in* $\mathrm{L}(E)$ *satisfying*

$$\tilde{\mathrm{e}}_{\mathrm{s}}(\lambda_n) \xrightarrow{\tau_{\mathrm{w}}} \mu \in M^1(E).$$

Then there exist measures $\lambda \in \mathrm{L}(E)$, $\varrho \in \mathrm{G}(E)$ *and an element* $z \in E$ *such that*

$$\mu = \tilde{\mathrm{e}}_{\mathrm{s}}(\lambda) * \varrho * \varepsilon_z .$$

In particular,

(i) *for every* $\delta \in C(\lambda)$ *we have that*

$$\lambda_n|^{\delta} \xrightarrow{\tau_{\mathrm{w}}} \lambda|^{\delta} ,$$

(ii)

$$-2 \operatorname{Log} \hat{\varrho}(a) = \lim_{\delta \downarrow 0} \limsup_{n \to \infty} \int_{U_{\delta}} \langle x, a \rangle^2 \, \lambda_n(\mathrm{d}x)$$

$$= \lim_{\delta \downarrow 0} \liminf_{n \to \infty} \int_{U_{\delta}} \langle x, a \rangle^2 \, \lambda_n(\mathrm{d}x)$$

holds for all $a \in E'$, *and*

(iii) z *admits a representation*

$$z = \int_{U_1 \cap \complement U_{\delta}} x \, \lambda(\mathrm{d}x) - \lim_{n \to \infty} \int_{U_1 \cap \complement U_{\delta}} x \, \lambda_n(\mathrm{d}x)$$

for $\delta \in C(\lambda) \cap \,]0, 1[$ *which also equals*

$$\lim_{\substack{\delta \downarrow 0 \\ \delta \in C(\lambda)}} \lim_{n \to \infty} \int_{U_{\delta} \cap \complement U_1} x \, \lambda_n(\mathrm{d}x) .$$

As a supplement we note that $z = 0$ *if* $1 \in C(\lambda)$.

Proof. 1. (Construction of λ). From the assumption we conclude that the sequence $(\tilde{e}_s(\lambda_n))_{n \geqslant 1}$ is τ_w-relatively compact, hence by Property 3.4.11.6 also $(\lambda_n|^\delta)_{n \geqslant 1}$ is τ_w-relatively compact (in $M^b(E)$). Next we show that there exist a subsequence $(\lambda_{n'}|^\delta)$ of $(\lambda_n|^\delta)_{n \geqslant 1}$ and a measure $\lambda \in M^\sigma(E)$ satisfying $\lambda(\{0\}) = 0$ and

$$\lambda_{n'}|^\delta \xrightarrow{\tau_w} \lambda|^\delta$$

for each $\delta \in C(\lambda)$. But then Theorem 3.4.3 implies that $(x(\lambda_n|^\delta))_{n \geqslant 1}$ is relatively compact whenever $\delta \in C(\lambda) \cap]0,1[$, and we may assume without loss of generality that

$$x(\lambda_{n'}|^\delta) \to x_\delta$$

for some $x_\delta \in E$. From Theorem 3.4.6 we now deduce that

$$e_s(\lambda_{n'}|^\delta) \xrightarrow{\tau_w} e_s(\lambda|^\delta) * \varepsilon_{z_\delta},$$

where

$$z_\delta := x_\delta - x(\lambda|^\delta)$$
$$= \lim_{n' \to \infty} \left(\int_{U_1 \cap \complement U_\delta} x \, \lambda(\mathrm{d}x) - \int_{U_1 \cap \complement U_\delta} x \, \lambda_{n'}(\mathrm{d}x) \right).$$

In view of Corollary 3.4.4 z_δ is independent of δ and hence can be named z. We obtain

$$e_s(\lambda_{n'}|^\delta) \xrightarrow{\tau_w} e_s(\lambda|^\delta) * \varepsilon_z$$

whenever $\delta \in C(\lambda) \cap]0,1[$. Since

$$e_s(\lambda_{n'}|^\delta) * e_s(\lambda_{n'}|_\delta) \xrightarrow{\tau_w} \mu,$$

there exists a measure $\varrho_\delta \in M^1(E)$ satisfying

$$e_s(\lambda_{n'}|_\delta) \xrightarrow{\tau_w} \varrho_\delta,$$

and from Corollary 2.2.4 we infer that

$$\mu = e_s(\lambda|^\delta) * \varrho_\delta * \varepsilon_z$$

as long as $\delta \in C(\lambda) \cap]0,1[$. For any sequence $(\delta_j)_{j \geqslant 1}$ in $C(\lambda)$ with $\delta_j \downarrow 0$ the sequence $(e_s(\lambda|^{\delta_j}))_{j \geqslant 1}$ is relatively shift compact by Theorem 2.2.7, hence τ_w-relatively compact by Property 3.4.11.5, and by the characterization theorem 3.4.9 we obtain that $\lambda \in L(E)$ and that

$$e_s(\lambda|^{\delta_j}) \xrightarrow{\tau_w} \tilde{e}_s(\lambda)$$

as $j \to \infty$. Applying Corollary 2.2.4 once more we achieve the τ_{w}-convergence of the sequence $(\varrho_{\delta_j})_{j \geqslant 1}$ towards a measure $\varrho \in M^1(E)$ and the representation

$$\mu = \tilde{\mathrm{e}}_{\mathrm{s}}(\lambda) * \varrho * \varepsilon_z .$$

In fact, since

$$\mathrm{e}_{\mathrm{s}}(\lambda_{n_j}|^{\delta_j}) \xrightarrow{\tau_{\mathrm{w}}} \varrho$$

as $j \to \infty$ for a suitable subsequence $(\lambda_{n_j}|^{\delta_j})$ we deduce from Theorem 3.4.17 that $\varrho \in G(E)$ and hence the desired representation of μ.

Now we show the remaining statements of the theorem.

(i). For an arbitrarily chosen $\delta \in C(\lambda)$ there exists a subsequence $(\lambda_{n_k})_{k \geqslant 1}$ of $(\lambda_n)_{n \geqslant 1}$ such that

$$\lambda_{n_k}|^{\delta} \xrightarrow{\tau_{\mathrm{w}}} \lambda_{\delta} \in M^{\mathrm{b}}(E)$$

as $k \to \infty$. From Property 3.4.11.6 we infer that $(\lambda_n|^{\delta})_{n \geqslant 1}$ is τ_{w}-relatively compact. Now we proceed in analogy to part 1. of this proof with $(\mathrm{e}_{\mathrm{s}}(\lambda_{n_k}))_{k \geqslant 1}$ in place of $(\mathrm{e}_{\mathrm{s}}(\lambda_n))_{n \geqslant 1}$ and achieve the representation

$$\mu = \tilde{\mathrm{e}}_{\mathrm{s}}(\lambda^{(1)}) * \varrho^{(1)} * \varepsilon_{z^{(1)}} .$$

But then the uniqueness theorem 3.4.15 implies that $\lambda^{(1)} = \lambda$, and again considering subsequences we obtain that $\lambda_{\delta} = \lambda|^{\delta}$ as the unique accumulation point of the τ_{w}-relatively compact sequence $(\lambda_n|^{\delta})_{n \geqslant 1}$.

(iii). The proof runs in analogy to that of (i). We fix $\delta \in C(\lambda) \cap \,]0, 1[$, observe that by Theorem 3.4.3 the sequence $(x(\lambda_n|^{\delta}) - x(\lambda^{\delta}))_{n \geqslant 1}$ is relatively compact in E, and passing to suitable subsequences together with an application of the uniqueness theorem 3.4.15 yields

$$\mu = \tilde{\mathrm{e}}_{\mathrm{s}}(\lambda) * \varrho * \varepsilon_z$$

with

$$z := \lim_{n \to \infty} x(\lambda_n|^{\delta}) - x(\lambda|^{\delta})$$

as asserted.

(ii). For each $\delta \in C(\lambda) \cap \,]0, 1[$ we have the limit relationships

$$\lambda_n|^{\delta} \xrightarrow{\tau_{\mathrm{w}}} \lambda|^{\delta}$$

and

$$x(\lambda_n|^{\delta}) \to z + x(\lambda|^{\delta}) .$$

Hence we apply Theorem 3.4.6 and obtain

$$\mathrm{e_s}(\lambda_n|^\delta) \xrightarrow{\ \tau_w\ } \mathrm{e_s}(\lambda|^\delta) * \varepsilon_z \,,$$

consequently with the help of Corollary 2.2.4

$$\mathrm{e_s}(\lambda_n|_\delta) \xrightarrow{\ \tau_w\ } \varrho_\delta \,,$$

where ϱ_δ has been introduced in part 1. of this proof. Then Corollary 2.1.16 yields

$$\mathrm{Log}\,\hat{\varrho}_\delta(a) = \lim_{n\to\infty} \int_{U_\delta} K(x,a)\,\lambda_n(\mathrm{d}x)$$

for all $a \in E'$. Given $\delta > 0$ and $a \in E'$ we choose a subsequence $(\lambda_{n_k})_{k\geqslant 1}$ of $(\lambda_n)_{n\geqslant 1}$ such that

$$\lim_{k\to\infty} \int_{U_\delta} \langle x,a\rangle^2\,\lambda_{n_k}(\mathrm{d}x) = \limsup_{n\to\infty} \int_{U_\delta} \langle x,a\rangle^2\,\lambda_n(\mathrm{d}x)\,.$$

Clearly,

$$r(\delta,a) := \lim_{k\to\infty} \int_{U_\delta} \left(K(x,a) + \frac{1}{2}\langle x,a\rangle \right) \lambda_{n_k}(\mathrm{d}x)$$

is a well-defined element of $\mathbb{C}$. But then

$$\mathrm{Log}\,\hat{\varrho}_\delta(a) = r(\delta,a) - \frac{1}{2}\limsup_{n\to\infty} \int_{U_\delta} \langle x,a\rangle^2\,\lambda_n(\mathrm{d}x)\,.$$

Moreover, employing Theorem 3.3.4 a Taylor expansion argument yields the estimate

$$|r(\delta,a)| \leqslant c(a)\delta$$

with a constant $c(a) \geqslant 0$, for all sufficiently small $\delta > 0$. Taking a sequence $(\delta_j)_{j\geqslant 1}$ in $C(\lambda)$ with $\delta_j > 0$ and $\delta_j \downarrow 0$ we obtain

$$\mathrm{Log}\,\hat{\varrho}(a) = \lim_{j\to\infty} \mathrm{Log}\,\hat{\varrho}_{\delta_j}(a)$$

$$= -\frac{1}{2}\lim_{j\to\infty}\limsup_{n\to\infty} \int_{U_{\delta_j}} \langle x,a\rangle^2\,\lambda_n(\mathrm{d}x)\,.$$

But obviously the function

$$\delta \mapsto \limsup_{n\to\infty} \int_{U_\delta} \langle x,a\rangle^2\,\lambda_n(\mathrm{d}x)$$

is increasing, thus $(\delta_j)_{j\geqslant 1}$ can be chosen in $\mathbb{R}$ arbitrarily, and the first equality in (ii) has been established. The second one is proved similarly. $\square$

In the special case of bounded Lévy measures the preceding theorem can be improved slightly.

Theorem 3.4.19 *Let $(\lambda_n)_{n\geqslant 1}$ be a sequence in $\mathrm{L}(E) \cap M^{\mathrm{b}}(E)$ such that*

$$\mathrm{e}(\lambda_n) \xrightarrow{\tau_{\mathrm{w}}} \mu \in M^1(E) \,.$$

Then there exist uniquely $\lambda \in \mathrm{L}(E)$, $\varrho \in \mathrm{G}(E)$ and $x_0 \in E$ satisfying

$$\mu = \tilde{\mathrm{e}}_{\mathrm{s}}(\lambda) * \varrho * \varepsilon_{x_0} \,.$$

Moreover, λ and ϱ admit the representations given in Theorem 3.4.18, and

$$x_0 = \lim_{\substack{\delta \downarrow 1 \\ \delta \in C(\lambda)}} \lim_{n \to \infty} \int_{U_\delta} x \, \lambda_n(\mathrm{d}x) \,.$$

Proof. Since

$$\mathrm{e}_{\mathrm{s}}(\lambda_n) * \varepsilon_{-x(\lambda_n)} \xrightarrow{\tau_{\mathrm{w}}} \mu \,,$$

the sequences $(\mathrm{e}_{\mathrm{s}}(\lambda_n))_{n\geqslant 1}$ and $(x(\lambda_n))$ are τ_{w}-relatively compact and relatively compact respectively. Indeed, one just applies Theorem 2.2.3 and Fact 3.4.5.1. Let $(\lambda_{n'})$ be a subsequence of (λ_n) such that

$$\mathrm{e}_{\mathrm{s}}(\lambda_{n'}) \xrightarrow{\tau_{\mathrm{w}}} \nu \in M^1(E) \,.$$

Then Theorem 3.4.6 implies that

$$x(\lambda_{n'}) \to z_0 \in E \,,$$

and one obtains the equality $\mu = \nu * \varepsilon_{-z_0}$. On the other hand we infer from Theorem 3.4.18 that

$$\mu = \tilde{\mathrm{e}}_{\mathrm{s}}(\lambda) * \varrho * \varepsilon_{z-z_0} \,,$$

where $\lambda \in L(E)$ with

$$\lambda|^\delta := \tau_{\mathrm{w}}- \lim_{n' \to \infty} \lambda_{n'}|^\delta$$

for $\delta \in C(\lambda)$ and

$$z := \lim_{n' \to \infty} x(\lambda_{n'}|^\delta) - x(\lambda|^\delta)$$

for $\delta \in C(\lambda) \cap \,]0, 1[$. By Theorem 3.3.3 the sequence $(\lambda_n|^\delta)_{n\geqslant 1}$ is τ_{w}-relatively compact, thus by the uniqueness theorem 3.4.15 we have

$$\lambda_n|^\delta \xrightarrow{\tau_{\mathrm{w}}} \lambda|^\delta$$

as $n \to \infty$ ((i) of Theorem 3.4.18).

Moreover, the sequence $(x(\lambda_n|^\delta))_{n\geqslant 1}$ is relatively compact by Theorem 3.4.3, hence the sequence $(x(\lambda_n|^\delta) - x(\lambda_n))_{n\geqslant 1}$ has the same property. Again applying the uniqueness theorem 3.4.15 we obtain

$$x_0 := z - z_0$$
$$= \lim_{n\to\infty} \left(x(\lambda_n|^\delta) - x(\lambda_n)\right) - x(\lambda|^\delta)$$

whenever $\delta \in C(\lambda) \cap \,]0, 1[$. An application of Corollary 3.4.4 provides us with the limit representation of x_0 ((iii) of Theorem 3.4.18).

Finally we prove the representation of $-\operatorname{Log} \hat{\varrho}$ ((ii) of Theorem 3.4.18).

Let $\delta \in C(\lambda) \cap \,]0, 1[$. By assumption

$$\mathrm{e}(\lambda_n|^\delta) * \mathrm{e}(\lambda_n|_\delta) \xrightarrow{\tau_\mathrm{w}} \mu \,.$$

But Theorem 2.4.12 (ii) implies that

$$\mathrm{e}(\lambda_n|^\delta) \xrightarrow{\tau_\mathrm{w}} \mathrm{e}(\lambda|^\delta) \,,$$

thus

$$\mathrm{e}(\lambda_n|_\delta) \xrightarrow{\tau_\mathrm{w}} \nu_\delta \in M^1(E)$$

and

$$\mu = \nu_\delta * \mathrm{e}(\lambda|^\delta)$$

(by Corollary 2.2.4). From

$$x(\lambda_n|_\delta) = x(\lambda_n) - x(\lambda_n|^\delta)$$

we conclude that

$$x(\lambda_n|_\delta) \to -x_0 - x(\lambda|^\delta)$$

hence that

$$\mathrm{e}_\mathrm{s}(\lambda_n|_\delta) \xrightarrow{\tau_\mathrm{w}} \nu_\delta * \varepsilon_{-x_0 - x(\lambda|^\delta)} \,.$$

Now Theorems 3.4.18 and 3.4.15 applied to $(\mathrm{e}_\mathrm{s}(\lambda_n|_\delta))_{n\geqslant 1}$ instead of $(\mathrm{e}_\mathrm{s}(\lambda_n))_{n\geqslant 1}$ yield the equality

$$\nu_\delta * \varepsilon_{-x_0 - x(\lambda|^\delta)} = \mathrm{e}_\mathrm{s}(\lambda|_\delta) * \varrho \,,$$

hence the assertion. □

Theorem 3.4.20 (Lévy–Khinchin decomposition of infinitely divisible measures)

Let $\mu \in \mathrm{I}(E)$ with the sequence $(\mu_{1/n})_{n\geqslant 1}$ of its n-th roots. Then there exist measures $\lambda \in \mathrm{L}(E)$, $\varrho \in \mathrm{G}(E)$ and an element x_0 of E such that

$$\mu = \tilde{\mathrm{e}}_\mathrm{s}(\lambda) * \varrho * \varepsilon_{x_0} \,.$$

Moreover, λ, ϱ and x_0 are uniquely determined and obtained as follows:

(i)
$$(n\mu_{1/n})|^\delta \xrightarrow{\ \tau_w\ } \lambda|^\delta$$

for all $\delta \in C(\lambda)$.

(ii)
$$-2\operatorname{Log}\hat{\varrho}(a) = \lim_{\delta\downarrow 0}\ \limsup_{n\to\infty}\ n\int_{U_\delta} \langle x,a\rangle^2\, \mu_{1/n}(\mathrm{d}x)$$

$$= \lim_{\delta\downarrow 0}\ \liminf_{n\to\infty}\ n\int_{U_\delta} \langle x,a\rangle^2\, \mu_{1/n}(\mathrm{d}x)$$

for all $a \in E'$.

(iii) $x_0 = \displaystyle\lim_{\substack{\delta\downarrow 1\\ \delta\in C(\lambda)}}\ \lim_{n\to\infty} n\int_{U_\delta} x\,\mu_{1/n}(\mathrm{d}x)$.

Proof. From the uniqueness of the roots $\mu_{1/n}$ of μ (Theorem 2.3.5) we infer that for every $n \in \mathbb{N}$ the measure

$$\lambda_n := n\big(\mu_{1/n} - \mu_{1/n}(\{0\})\varepsilon_0\big)$$

belongs to $L(E)$. But Property 2.4.10.2 tells us that

$$\mathrm{e_s}(n\mu_{1/n}) = \mathrm{e_s}(\lambda_n) * \mathrm{e_s}\big(n\mu_{1/n}(\{0\})\varepsilon_0\big) = \mathrm{e_s}(\lambda_n)$$

for all $n \in \mathbb{N}$, and Theorem 2.4.11 implies that

$$\mathrm{e_s}(\lambda_n) \xrightarrow{\ \tau_w\ } \mu\,.$$

In view of

$$\lambda_n|^\delta = (n\mu_{1/n})|^\delta\,,$$

$$\int_{U_\delta} \langle x,a\rangle^2\, \lambda_n(\mathrm{d}x) = n\int_{U_\delta} \langle x,a\rangle^2\, \mu_{1/n}(\mathrm{d}x)\,,$$

and

$$\int_{U_\delta} x\,\lambda_n(\mathrm{d}x) = n\int_{U_\delta} x\,\mu_{1/n}(\mathrm{d}x)$$

valid for all $a \in E'$, $\delta \in \mathbb{R}_+^\times$ $(n \in \mathbb{N})$ Theorem 3.4.19 implies the assertion. $\square$

Remark 3.4.21 In short Theorem 3.4.20 says that any $\mu \in \mathrm{I}(E)$ admits a **Lévy–Khinchin representation** of the form

$$\hat{\mu}(a) = \exp\left\{\mathrm{i}\langle x_0,a\rangle - \frac{1}{2}\langle Ra,a\rangle + \int \left(\mathrm{e}^{\mathrm{i}\langle x,y\rangle} - 1 - \mathrm{i}\langle x,a\rangle\mathbb{1}_{U_1}(x)\right)\lambda(\mathrm{d}x)\right\}$$

valid for all $a \in E'$, where $x_0 \in E$, R is a symmetric linear mapping $E' \to E$ and λ a Lévy measure.

One observes that μ is symmetric if and only if λ is symmetric, and in this case $x_0 = 0$.

On the other hand, if λ satisfies the condition

$$\int_{U_1} \|x\|^2 \, \lambda(\mathrm{d}x) < \infty \,,$$

then the kernel

$$(x, a) \mapsto \mathrm{e}^{\mathrm{i}\langle x, a\rangle} - 1 - \mathrm{i}\langle x, a\rangle \mathbb{1}_{U_1}$$

(formerly abbreviated by K) can be replaced by the classical more familiar one

$$(x, a) \mapsto \mathrm{e}^{\mathrm{i}\langle x, a\rangle} - 1 - \frac{\mathrm{i}\langle x, a\rangle}{1 + \|x\|^2} \,.$$

In this case the measure $\nu \in \mathrm{I}(E)$ given by

$$\hat{\nu}(a) := \exp\left\{ -\int \left(\mathrm{e}^{\mathrm{i}\langle x, a\rangle} - 1 - \frac{\mathrm{i}\langle x, a\rangle}{1 + \|x\|^2} \right) \lambda(\mathrm{d}x) \right\}$$

for all $a \in E'$ is a translate of $\tilde{\mathrm{e}}_{\mathrm{s}}(\lambda)$.

Chapter 4

Harmonic Analysis of Convolution Semigroups

4.1   Convolution of Radon measures

We start with an adaptation of the concept of measure given in Chapter 1 to locally compact spaces E. **Complex Borel measures** μ on E are introduced as complex-valued σ-additive set functions μ on the Borel-σ-algebra $\mathfrak{B}(E)$ of E having the property that $\mu(B)$ is finite for each relatively compact subset B of E. The totality of complex Borel measures will be abbreviated by $M_0(E)$. Notice that measures in $M_{0,+}(G)$ may take on the value ∞. For every $\mu \in M_0(E)$ one introduces the total variation $|\mu|$ of μ by

$$|\mu|(B) := \sup \left\{ \sum_{i \in I} |\mu(B_i)| : |I| < \infty, \ B_i \in \mathfrak{B}(E) \right.$$

$$\left. \text{for all } i \in I, \ \bigcup_{i \in I} B_i = B \right\},$$

whenever $B \in \mathfrak{B}(E)$. It is shown that $|\mu| \in M_0(E)$ and, of course, $|\mu| \geqslant 0$. $\mu \in M_0(E)$ is said to be regular if for every $B \in \mathfrak{B}(E)$

$$|\mu|(B) = \sup\{|\mu|(K) : K \in \mathcal{K}(E), \ K \subset B\}$$
$$= \inf\{|\mu|(O) : O \in \mathcal{O}(E), \ O \supset B\}.$$

The symbol $M(E)$ will serve as a short form for the set of all regular complex Borel measures in E. Finally, for $\mu \in M_0(E)$ we set

$$\|\mu\| := |\mu|(E)$$

and recognize that the set

$$M^{\mathrm{b}}(E) := \{\mu \in M(E) : \|\mu\| < \infty\}$$

of all **finite regular** (complex Borel) **measures** in E forms a normed vector space over $\mathbb{C}$, with $(\mu, \nu) \mapsto \mu + \nu$ and $\mu \mapsto \alpha\mu$ as vector space operations and $\| \cdot \|$ as the underlying norm. The following sequence of implications

$$M^1(E) \subset M_+^{\mathrm{b}}(E) \subset M_{\mathbb{R}}^{\mathrm{b}}(E) \subset M_{\mathbb{C}}^{\mathrm{b}}(E) := M^{\mathrm{b}}(E)$$

starting with the set

$$M^1(E) := \{\mu \in M_+^{\mathrm{b}}(E) : \|\mu\| = 1\}$$

of **probability measures** speaks for itself. As for the function spaces applied in the sequel we have the sequence of implications of vector spaces

$$C^{\mathrm{c}}(E) \subset C^0(E) \subset C^{\mathrm{b}}(E) \subset C(E) \, ,$$

where the corresponding symbols stand for the complex continuous functions on E which have compact support, vanish at infinity, are bounded and just continuous respectively. While $C(E)$ carries the topology τ_{co} of compact convergence and $C^{\mathrm{c}}(E)$ the canonical inductive limit topology, $C^0(E)$ and $C^{\mathrm{b}}(E)$ are furnished with the topology of uniform convergence. We note that $C^{\mathrm{c}}(E)^- = C^0(E)$. For any vector space B over $\mathbb{C}$ the symbols $L^{\mathrm{b}}(B)$ or $L_+(B)$ will denote the sets of bounded or positive linear functionals on B respectively.

In the above described extended set-up of measure theory the *Riesz representation theorem* takes on a more general form valid beyond the metric case.

Theorem 4.1.1 *There exists an isometric isomorphism*

$$\mu \mapsto L_\mu$$

from $M^{\mathrm{b}}(E)$ *onto* $C^0(E)^* := L^{\mathrm{b}}(C^0(E))$ *given by*

$$L_\mu(f) := \int f \, \mathrm{d}\mu$$

for all $f \in C^0(E)$.

In fact, to every $L \in L^{\mathrm{b}}(C^0(E))$ *there corresponds a unique* $\mu \in M^{\mathrm{b}}(E)$ *satisfying* $L = L_\mu$, *and*

$$\sup\{|L(f)| : f \in C^0(E), \|f\| = 1\} = \|\mu\| \, .$$

In particular, $(M^{\mathrm{b}}(E), \| \cdot \|)$ *is a Banach space over* $\mathbb{C}$.

It should be noted that a similar correspondence is available for the sets $L_+(C^{\mathrm{c}}(E))$ *and* $M_+(E)$ *instead of* $L^{\mathrm{b}}(C^0(E))$ *and* $M^{\mathrm{b}}(E)$ *respectively.*

This correspondence justifies Bourbaki's introduction of measures on locally compact spaces E as continuous linear functionals on $C^{\mathrm{c}}(E)$. In other words, $M(E)$ will be interpreted as the topological dual $C^{\mathrm{c}}(E)'$ of $C^{\mathrm{c}}(E)$, and $M(E)$ will appear as the set of **Radon measures** on E.

Definition 4.1.2 For any $\nu \in M_{0,+}(E)$ and $p \in]0, \infty[$ one introduces the Lebesgue space $L^p(E, \nu)$ of (classes of) p-times ν-integrable complex-valued functions on E and notes that $L^p(E, \nu)$ is a Banach space provided $p \geqslant 1$, a Hilbert space for $p = 2$, and that
$$L^p(E, \nu) = C^{\mathrm{c}}(E)^-$$
if $\nu \in M_+(E)$, i.e. if ν is regular. Here the closure of $C^{\mathrm{c}}(E)$ is taken in the p-norm topology of $L^p(E, \nu)$.

For $\mu \in M^{\mathrm{b}}(E)$ and $\nu \in M_{0,+}(E)$ the Radon-Nikodym equivalence is available: μ is ν-(absolutely) continuous if and only if there exists a function $f \in L^1(E, \nu)$ such that $\mu = f \cdot \nu$. In particular, $L^1(E, \nu)$ is isometrically embedded into $M^{\mathrm{b}}(E)$, i.e.
$$\|\mu\| = \|f\|,$$
for every $\mu \in M^{\mathrm{b}}(E)$ of the form $\mu := f \cdot \nu$ with $f \in L^1(E, \nu)$.

On $M(E)$ the **vague topology** τ_{v} is introduced as the topology $\sigma(C^{\mathrm{c}}(E)', C^{\mathrm{c}}(E))$ for the dual pair $(C^{\mathrm{c}}(E)', C^{\mathrm{c}}(E))$ arising from the normed vector space $C^{\mathrm{c}}(E)$. On $M^{\mathrm{b}}(E)$ the vague topology can be compared with the **weak topology** considered as the topology $\sigma(C^{\mathrm{b}}(E)', C^{\mathrm{b}}(E))$ for the dual pair arising from $C^{\mathrm{b}}(E)$. In fact, τ_{w} is finer than τ_{v}.

Proposition 4.1.3 *For any net* $(\mu_\alpha)_{\alpha \in A}$ *in* $M_+^{\mathrm{b}}(E)$ *and any measure* $\mu \in M_+^{\mathrm{b}}(E)$ *the following statements are equivalent:*

(i) $\tau_{\mathrm{w}} - \lim_\alpha \mu_\alpha = \mu$.
(ii) $\tau_{\mathrm{v}} - \lim_\alpha \mu_\alpha = \mu$ *and* $\lim_\alpha \|\mu_\alpha\| = \|\mu\|$.

Proof. It suffices to show the implication (ii) $\Rightarrow$ (i). Let (ii) be satisfied for a net $(\mu_\alpha)_{\alpha \in A}$ and a measure μ in $M_+^{\mathrm{b}}(E)$. We take a function $f \in C^{\mathrm{b}}(E)$ and fix $\varepsilon > 0$. Since μ is regular, there exists a set $K \in \mathcal{K}(E)$ such that $\mu(\complement K) < \varepsilon$. Now choose a function $g \in C^{\mathrm{c}}(E)$ satisfying $0 \leqslant g \leqslant 1$ and $g(x) = 1$ for all $x \in K$. By assumption we have
$$\lim_\alpha \int (1 - g) \, \mathrm{d}\mu_\alpha = \int (1 - g) \, \mathrm{d}\mu$$
$$\leqslant \mu(\complement K) < \varepsilon$$

and
$$\lim_\alpha \int fg \, d\mu_\alpha = \int fg \, d\mu \,.$$
Consequently, there exists an $\alpha_0 \in A$ such that for all $\alpha \in A$ with $\alpha \geqslant \alpha_0$ the inequalities
$$\int (1-g) d\mu_\alpha < \varepsilon$$
and
$$\left| \int fg \, d\mu_\alpha - \int fg \, d\mu \right| < \varepsilon$$
hold. But this implies for all $\alpha \in A$ with $\alpha \geqslant \alpha_0$ that
$$\left| \int f \, d\mu_\alpha - \int f \, d\mu \right| \leqslant \left| \int fg \, d\mu_\alpha - \int fg \, d\mu \right| + \left| \int f(1-g) \, d\mu_\alpha \right|$$
$$+ \left| \int f(1-g) \, d\mu \right|$$
$$\leqslant \varepsilon(1 + 2\|f\|) \,,$$
hence that
$$\tau_{\mathrm{w}} - \lim_\alpha \mu_\alpha = \mu \,. \qquad \square$$

It follows from Proposition 4.1.3 that on $M^1(E)$ the topologies τ_{w} and τ_{v} coincide.

Further topological properties concern the τ_{v}-metrizability of $M_+(E)$ which holds if and only if E admits a countable basis of its topology, and the τ_{v}-compactness of $M^1(E)$ which is equivalent to the compactness of E.

From now on let $E := G$ be a locally compact Abelian **group**. For every $a \in G$ the group translation $x \mapsto x+a$ can be extended to functions f on G by
$$T_a f(x) := f_a(x) := f(x-a)$$
for all $x \in G$ and to measures μ on G by
$$T_a(\mu)(f) := \int f_{-a}(x) \, \mu(dx)$$
$$= \int f(x+a) \, \mu(dx)$$
for all $f \in C^{\mathrm{c}}(G)$.

Definition 4.1.4 A measure $\mu \in M_+(G)$ is called a **Haar measure** on G if $\mu \neq 0$ and if μ is **translation invariant** in the sense that
$$\mu(T_a f) = \mu(f)$$
for all $f \in C^{\mathrm{c}}(G)$ and $a \in G$.

Theorem 4.1.5 *On any locally compact Abelian group G there exists a Haar measure.*

Proof.   1.   Producing the crucial functional.
In the following the space $C^c(G)$ is assumed to contain only functions $\neq 0$.   For $f, \varphi \in C_+^c(G)$ we define

$$(f : \varphi) := \inf \left\{ \sum_{j=1}^{n} c_j : f \leqslant \sum_{j=1}^{n} c_j\, \varphi_{x_j} \text{ for } c_1, \ldots, c_n \in \mathbb{R}_+ , \right.$$

$$\left. x_1, \ldots, x_n \in G, \ n \geqslant 1 \right\}$$

and derive the following properties

(1) $(f : \varphi) = (T_y f : \varphi)$ for all $y \in G$.
(2) $(f_1 + f_2 : \varphi) \leqslant (f_1 : \varphi) + (f_2 : \varphi)$.
(3) $(cf : \varphi) = c(f : \varphi)$ for any $c \in \mathbb{R}_+^\times$.
(4) $(f_1 : \varphi) \leqslant (f_2 : \varphi)$ whenever $f_1 \leqslant f_2$.
(5) $(f : \varphi) \geqslant \frac{\|f\|}{\|\varphi\|}$.
(6) $(f : \varphi) \leqslant (f : \psi)(\psi : \varphi)$ for any $\psi \in C_+^c(G)$.

In these statements f, f_1, f_2 and φ are functions in $C_+^c(G)$.
    In order to justify the above assertions it suffices to note that since the compact support of f can be covered by a finite number N of translates of the set

$$\left\{ x \in G : \varphi(x) > \frac{1}{2} \|\varphi\| \right\},$$

$$(f : \varphi) \leqslant 2N \frac{\|f\|}{\|\varphi\|}$$

and hence $(f : \varphi)$ is well-defined. While properties (1) to (5) are easily verified, property (6) follows from the additional observation that for

$$f \leqslant \sum_{j=1}^{n} c_j\, \psi_{x_j}$$

and

$$\psi \leqslant \sum_{k=1}^{m} d_k\, \varphi_{y_k}$$

with the obvious meaning of the summands involved,

$$f \leqslant \sum_{j=1}^{n} \sum_{k=1}^{m} c_j\, d_k\, \varphi_{x_j + y_k}$$

holds.

Now, we fix $f_0 \in C_+^c(G)$ and define

$$I_\varphi(f) := \frac{(f : \varphi)}{(f_0 : \varphi)}$$

whenever $f, \varphi \in C_+^c(G)$. From properties (1) to (4) we deduce that I_φ is invariant, subadditive, homogeneous, and increasing. Moreover, by property (6) we obtain that

(7) $\frac{1}{(f_0 : f)} \leqslant I_\varphi(f) \leqslant (f : f_0)$.

2. A preparative inequality.

We show that given $f_1, f_2 \in C_+^c(G)$ and $\varepsilon > 0$ there exists a $V \in \mathfrak{V}(0)$ such that

$$I_\varphi(f_1) + I_\varphi(f_2) \leqslant I_\varphi(f_1 + f_2) + \varepsilon$$

whenever $\operatorname{supp}(\varphi) \subset V$.

In fact, let $g \in C_+^c(G)$ such that $g(x) = 1$ for all $x \in \operatorname{supp}(f_1 + f_2)$, and let $\delta > 0$. Moreover, let $h := f_1 + f_2 + \delta g$ and for $i = 1, 2$

$$h_i = \frac{f_i}{h}$$

with the convention that $h_i = 0$ whenever $h = 0$. Then $h_i \in C_+^c(G)$, hence there exists a $V \in \mathfrak{V}(0)$ such that

$$|h_i(x) - h_i(y)| < \delta$$

for $x - y \in V$ $(i = 1, 2)$. Now we take $\varphi \in C_+^c(G)$ with $\operatorname{supp}(\varphi) \subset V$. If

$$h \leqslant \sum_{j=1}^{n} c_j \, \varphi_{x_j}$$

with the summands as described above, then

$$f_i(x) = h(x)h_i(x)$$

$$\leqslant \sum_{j=1}^{n} c_j \varphi(x - x_j) h_i(x)$$

$$\leqslant \sum_{j=1}^{n} c_j \varphi(x - x_j)(h_i(x_j) + \delta) \,,$$

since $|h_i(x) - h_i(x_j)| < \delta$ whenever $x - x_j \in \text{supp}(\varphi)$. But we have $h_1 + h_2 \leqslant 1$, hence obtain

$$(f_1 : \varphi) + (f_2 : \varphi) \leqslant \sum_{j=1}^{n} c_j (h_1(x_j) + \delta) + \sum_{j=1}^{n} c_j (h_2(x_j) + \delta)$$
$$\leqslant \sum_{j=1}^{n} c_j (1 + 2\delta),$$

and taking the infimum over all sums of the form $\sum_{j=1}^{n} c_j$, properties (2) and (3) imply that

$$I_\varphi(f_1) + I_\varphi(f_2) \leqslant (1 + 2\delta) I_\varphi(h)$$
$$\leqslant (1 + 2\delta)\big(I_\varphi(f_1 + f_2) + \delta I_\varphi(g)\big)$$

holds. Finally, property (7) yields

$$2\delta(f_1 + f_2 : f_0) + \delta(1 + 2\delta)(g : f_0) < \varepsilon$$

once δ is chosen sufficiently small, and this proves the assertion.

3. The completion of the proof.

For each $f \in C_+^c(G)$ let M_f denote the interval $\left[\frac{1}{(f_0:f)}, (f : f_0)\right]$ arising from property (7), and put $M := \Pi_{f \in C_+^c(G)} M_f$. Clearly, M is a compact space which by property (7) contains all mappings I_φ for $\varphi \in C_+^c(G)$. For each $V \in \mathfrak{V}(0)$ let $c(V)$ denote the closure in M of the set of those I_φ for which $\text{supp}(\varphi) \subset V$. From

$$\bigcap_{j=1}^{n} c(V_j) \supset c\left(\bigcap_{j=1}^{n} V_j\right)$$

for neighborhoods $V_j \in \mathfrak{V}(0)$ we conclude that the system of sets $c(V)$ $(V \in \mathfrak{V}(0))$ has the finite intersection property, hence the compactness of M secures the existence of an element I of M with $I \in c(V)$ for all $V \in \mathfrak{V}(0)$ which implies that every neighborhood of I in M contains mappings I_φ with arbitrarily small $\text{supp}(\varphi)$. This means that for any $V \in \mathfrak{V}(0)$, any $\varepsilon > 0$, and all $f_1, \ldots, f_n \in C_+^c(G)$ there exists a $\varphi \in C_+^c(G)$ satisfying $\text{supp}(\varphi) \subset V$ and

$$|I(f_j) - I_\varphi(f_j)| < \varepsilon$$

for all $j = 1, \ldots, n$. From properties (1) to (3) and part 2. of this proof we deduce that I is translation invariant, additive and homogeneous.

Now, any $f \in C_+^c(G)$ admits a representation $f = g - h$ with $g, h \in C_+^c(G)$. If, in addition $f = g' - h'$ with $g', h' \in C_+^c(G)$, then $g + h' = h + g'$, hence

$$I(g) + I(h') = I(h) + I(g'),$$

and

$$I(f) := I(g) - I(h)$$

yields a well-defined extension of I to a positive linear functional on $C^c(G)$ which by the Riesz Theorem 4.1.1 is a Haar measure. $\qquad\square$

Theorem 4.1.6 *For two Haar measures μ and ν on G there exists a constant $c \in \mathbb{R}_+^\times$ such that*

$$\nu = c\,\mu.$$

Proof. Let $g \in C^c(G)$ with $\int g \,\mathrm{d}\mu = 1$. Then, putting

$$c := \int_G g(-x)\,\nu(\mathrm{d}x)$$

we obtain for any $f \in C^c(G)$ that

$$\int_G f \,\mathrm{d}\nu = \int_G g(y)\,\mu(\mathrm{d}y) \int_G f(x)\,\nu(\mathrm{d}x)$$

$$= \int_G g(y)\,\mu(\mathrm{d}y) \int_G f(x+y)\,\nu(\mathrm{d}x)$$

$$= \int_G \left(\int_G g(y)f(x+y)\,\mu(\mathrm{d}y) \right) \nu(\mathrm{d}x)$$

$$= \int_G \left(\int_G g(y-x)f(y)\,\mu(\mathrm{d}y) \right) \nu(\mathrm{d}x)$$

$$= \int_G f(y)\,\mu(\mathrm{d}y) \int_G g(y-x)\,\nu(\mathrm{d}x)$$

$$= c \int_G f \,\mathrm{d}\mu,$$

hence that $\nu = c\,\mu$. In the above chain of equalities the Fubini theorem was applicable since the integrands of the double integrals belong to $C^c(G \times G)$. $\qquad\square$

Convention 4.1.7 Since by Theorem 4.1.6 Haar measure is unique up to a multiplicative constant, one talks about **the** Haar measure of G and denotes it by $\omega = \omega_G$.

Properties 4.1.8

4.1.8.1 ω *is positive on non-empty open subsets of* G, *i.e.* $\omega(O) > 0$ *for all* $O \in \mathcal{O}(G)$, $O \neq \emptyset$.

4.1.8.2 ω *is **inverse invariant**, i.e.* $\omega(-B) = \omega(B)$ *for all* $B \in \mathfrak{B}(G)$.

Definition 4.1.9 A pair (μ, ν) of measures in $M(G)$ is said to be **convolvable** if the integral

$$\int_{G \times G} f(x + y) \, \mu \otimes \nu(\mathrm{d}(x, y))$$

exists in $\mathbb{C}$ for every $f \in C^{\mathrm{c}}(G)$. In this case the mapping

$$f \mapsto \int_{G \times G} f(x + y) \, \mu \otimes \nu(\mathrm{d}(x, y))$$

is a continuous linear functional on $C^{\mathrm{c}}(G)$, hence a measure in $M(G)$ ($\cong C^{\mathrm{c}}(G)'$); it is called the **convolution** of μ and ν and is denoted by $\mu * \nu$.

In the case of convolvability the convolution viewed as a mapping

$$(\mu, \nu) \mapsto \mu * \nu$$

from $M(G) \times M(G)$ into $M(G)$ yields a commutative and associative operation.

The following result subsuming various useful properties of the convolution is easily proved.

Theorem 4.1.10 *Any pair* (μ, ν) *in* $M^{\mathrm{b}}(G) \times M^{\mathrm{b}}(G)$ *is convolvable, and together with convolution* $\mu * \nu$ *and norm* $\| \cdot \|$ *the space* $M^{\mathrm{b}}(G)$ *becomes a commutative Banach algebra with* ε_0 *as (convolution) unit element. In particular,*

$$\|\mu * \nu\| \leqslant \|\mu\| \, \|\nu\|$$

whenever $\mu, \nu \in M^{\mathrm{b}}(G)$.

Moreover, the Banach algebra $M^{\mathrm{b}}(G)$ *is involutive with respect to the involution* $\mu \mapsto \mu^{\sim}$ *given by*

$$\mu^{\sim}(f) := \overline{\mu(f^{\sim})}$$

for all $f \in C^{\mathrm{c}}(G)$, *where* $f^{\sim} = \overline{f^*}$ *with* $f^*(x) := f(-x)$ *for all* $x \in G$. *Here, the properties defining the involution read as follows*

$$\mu^{\sim\sim} = \mu \,,$$

$$(\mu * \nu)^{\sim} = \nu^{\sim} * \mu^{\sim}, \ and$$

$$\|\mu^{\sim}\| = \|\mu\|,$$

whenever $\mu, \nu \in M^{\mathrm{b}}(G)$.

For measures $\mu, \nu \in M^{\mathrm{b}}_{+}(G)$ the **support formula**

$$\mathrm{supp}(\mu * \nu) = (\mathrm{supp}(\mu) + \mathrm{supp}(\nu))^{-}$$

holds.

Definition 4.1.11 Applying the convolution to measures of the form $\mu := f \cdot \omega_G$ for $f \in L^1(G, \omega_G)$ one obtains a convolution in $L^1(G, \omega_G)$ which for $f, g \in L^1(G, \omega_G)$ is given by

$$f * g(x) = \int f(x - y)g(y)\, \omega_G(\mathrm{d}y)$$

whenever $x \in G$.

Clearly, together with this convolution and the norm $\|\cdot\|_1$ the space $L^1(G, \omega_G)$ becomes a closed ideal of $M^{\mathrm{b}}(G)$ and therefore also a commutative Banach algebra with involution.

$M^{\mathrm{b}}(G)$ and $L^1(G, \omega_G)$ are called the **measure algebra** and the **group algebra** of G respectively.

While $M^{\mathrm{b}}(G)$ has a unit ε_0, $L^1(G, \omega_G)$ has a unit only if G is discrete.

For functions f, g in an arbitrary Lebesgue space $L^p(G, \omega_G)$ ($p \in [1, \infty]$) the convolution $f * g$ is introduced by

$$f * g(x) := \int_G f(x - y)g(y)\, \omega_G(\mathrm{d}y)$$

provided that

$$\int_G |f(x - y)g(y)|\, \omega_G(\mathrm{d}y) < \infty$$

for all $x \in G$.

Proposition 4.1.12 *For each* $f \in L^p(G, \omega_G)$ ($p \in [1, \infty[$) *the mapping* $x \mapsto f_x$ *from* G *into* $L^p(G, \omega_G)$ *is uniformly continuous.*

Proof. Let $f \in L^p(G, \omega_G)$ and let $\varepsilon > 0$. Since $C^{\mathrm{c}}(G)$ is dense in $L^p(G, \omega_G)$, there exists a function $g \in C^{\mathrm{c}}(G)$ such that

$$\|g - f\|_p < \frac{\varepsilon}{3}.$$

Let $K := \operatorname{supp}(g)$. From the uniform continuity of g we deduce the existence of a neighborhood $V \in \mathfrak{V}(0)$ such that

$$\|g - g_x\| < \frac{\varepsilon}{3\omega_G(K)^{1/p}}$$

for all $x \in V$. Therefore

$$\|g - g_x\|_p < \frac{\varepsilon}{3}$$

and hence

$$\|f - f_x\|_p \leqslant \|f - g\|_p + \|g - g_x\|_p + \|g_x - f_x\|_p < \varepsilon$$

whenever $x \in V$. Finally we note that $f_x - f_y = (f - f_{y-x})_x$, so that

$$\|f_x - f_y\|_p < \varepsilon$$

whenever $y - x \in V$. $\qquad\square$

Proposition 4.1.13 *Let $\mathfrak{U}$ be a neighborhood system of $0 \in G$. For each $U \in \mathfrak{U}$ let ψ_U be a measurable function on G with compact support $\operatorname{supp}(\psi_U) \subset U$ such that $\psi_U \geqslant 0$, $\psi_U^* = \psi_U$ and $\int \psi_U \, d\omega_G = 1$. Then*

$$\|f * \psi_U - f\|_p \to 0 \quad as \quad U \to \{0\}$$

for all $f \in L^p(G, \omega_G)$, $p \in [1, \infty[$.

The family $\{\psi_U : U \in \mathfrak{U}\}$ is said to be an **approximate identity** in $L^p(G, \omega_G)$.

Proof. By Proposition 4.1.12 we choose $U \in \mathfrak{U}$ such that

$$\|f - f_y\|_p \to 0 \quad as \quad U \to \{0\}$$

whenever $y \in G$. But for each $h \in L^q(G, \omega_G)$ (where q is conjugate to p) Fubini's theorem and Hölder's inequality imply

$$\left| \int\limits_G (f * \psi_U - f) h \, d\omega_G \right| \leqslant \|h\|_q \int\limits_G \|f - f_y\|_p \psi_U(y) \, \omega_G(dy)$$

and therefore

$$\|f * \psi_U - f\|_p \leqslant \int_U \|f - f_y\|_p \psi_U(y) \, \omega_G(dy)$$

$$\leqslant \sup_{y \in U} \|f - f_y\|_p \to 0 \quad as \quad U \to \{0\}.$$

$\qquad\square$

4.2   Duality of locally compact Abelian groups

Generalizing classical Fourier theory for Euclidean spaces, two settings can be chosen in order to establish the appropriate analysis: locally convex vector spaces E, the structure underlying Chapters 1 to 3, and locally compact Abelian groups G, the structure to remain the basis of discussion in the subsequent chapters 4 to 6. The counterpart of the linear dual of E will be the group of continuous characters of G.

Definition 4.2.1 Any homomorphism $\chi : G \to \mathbb{T}$ is called a **character** of G.

The set $G^\wedge$ of all continuous characters of G forms a group under addition in the sense that for $\chi, \rho \in G^\wedge$ the sum is given by

$$(\chi + \rho)(x) := \chi(x)\,\rho(x)\,,$$

the inverse $-\chi$ of χ by

$$(-\chi)(x) := \overline{\chi(x)}\,,$$

and the neutral element 0 by

$$0(x) := 1$$

whenever $x \in G$.

In view of the duality between G and $G^\wedge$ to be discussed at a later stage, $G^\wedge$ is called the **dual** of G.

In the sequel we shall show that $G^\wedge$ can be furnished with a topology that makes it a locally compact group so that G and $G^\wedge$ belong to the same category of objects and consequently $G^{\wedge\wedge} := (G^\wedge)^\wedge$ can be formed.

Theorem 4.2.2 *There exists a one-to-one correspondence between the dual $G^\wedge$ of G and the space $M(L^1(G, \omega_G))$ of all nonvanishing multiplicative linear functionals on the group algebra $L^1(G, \omega_G)$ of G given as a mapping*

$$\chi \mapsto \tau_\chi$$

with

$$\tau_\chi(f) := \int_G f\overline{\chi}\,\mathrm{d}\omega_G$$

for all $f \in L^1(G, \omega_G)$.

Proof. 1. The mapping $f \mapsto \tau_\chi(f)$ is obviously linear. Moreover it is multiplicative, as follows from the following sequence of equalities valid for all $f, g \in L^1(G, \omega_G)$ and $\chi \in G^\wedge$:

$$\tau_\chi(f * g) = \int (f * g)(x)\,\overline{\chi(x)}\,\omega_G(\mathrm{d}x)$$

$$= \int \overline{\chi(x)}\,\omega_G(\mathrm{d}x) \int f_y(x)\,g(y)\,\omega_G(\mathrm{d}y)$$

$$= \int g(y)\,\overline{\chi(y)}\,\omega_G(\mathrm{d}y) \int f_y(x)\,\chi(-x+y)\,\omega_G(\mathrm{d}x)$$

$$= \tau_\chi(f)\,\tau_\chi(g)\,.$$

The nonvanishing of the mapping $f \mapsto \tau_\chi(f)$ follows from $\tau_\chi(f) \neq 0$ for some $f \in L^1(G, \omega_G)$, since $|\chi(x)| = 1$ for all $x \in G$. One concludes that $\tau_\chi \in M(L^1(G, \omega_G))$.

2. Now, let $\tau \in M(L^1(G, \omega_G))$. Since $L^1(G, \omega_G) \cong L^\infty(G, \omega_G)^*$ and τ is a bounded linear functional with $\|\tau\| = 1$, there exists a function $\varphi \in L^\infty(G, \omega_G)$ with $\|\varphi\| = 1$ such that

$$\tau(f) = \int_G f\varphi \,\mathrm{d}\omega_G$$

for all $f \in L^1(G, \omega_G)$. For $f, g \in L^1(G, \omega_G)$ we have

$$\int \tau(f)g\varphi \,\mathrm{d}\omega_G = \tau(f)\,\tau(g)$$

$$= \tau(f * g) = \int (f * g)\varphi \,\mathrm{d}\omega_G$$

$$= \int g \,\mathrm{d}\omega_G \int f_y(x)\varphi(x)\,\omega_G(\mathrm{d}x)$$

$$= \int g(y)\tau(f_y)\,\omega_G(\mathrm{d}y)\,,$$

consequently

$$\tau(f)\,\varphi(y) = \tau(f_y) \tag{1}$$

for ω_G-a.a. $y \in G$. It follows from Proposition 4.1.12 that the continuity of τ implies the continuity of $y \mapsto \tau(f_y)$ for each $f \in L^1(G, \omega_G)$. Choosing $f \in L^1(G, \omega_G)$ such that $\tau(f) \neq 0$ we deduce from (1) that φ is ω_G-a.e. continuous. Hence φ can be assumed to be continuous, and (1) holds for all $y \in G$.

Now we replace y by $x + y$ and then f by f_x in (1) in order to obtain

$$\tau(f)\,\varphi(x + y) = \tau(f_{x+y}) = \tau((f_x)_y)$$
$$= \tau(f_x)\,\varphi(y) = \tau(f)\,\varphi(x)\,\varphi(y)$$

and consequently

$$\varphi(x + y) = \varphi(x)\,\varphi(y)$$

for all $x, y \in G$. In particular, $\varphi^* = \varphi^{-1}$. Since $|\varphi| \leqslant 1$ it follows that $|\varphi| = 1$, hence that $\varphi \in G^\wedge$ and so $\tau = \tau_\varphi$.

3. As for the uniqueness of φ we just note that

$$\tau_\chi(f) = \tau_\rho(f)$$

for $\chi, \rho \in G^\wedge$ and all $f \in L^1(G, \omega_G)$ implies that

$$\chi(x) = \rho(x)$$

for ω_G-a.a. $x \in G$, but since χ, ρ are continuous functions, even for all $x \in G$. $\qquad\square$

For every $f \in L^1(G, \omega_G)$ the function $\hat{f} = \mathcal{F}(f)$ defined by

$$\hat{f}(\chi) := \tau_\chi(f)$$

for all $\chi \in G^\wedge$ is said to be the **Fourier transform** of f.

In terms of Gelfand's theory as described in Appendix C $\hat{f}$ is the Gelfand transform of f and $f \mapsto \hat{f}$ the Gelfand mapping $\mathcal{F}$ on $L^1(G, \omega_G)$. The space $M(L^1(G, \omega_G))$ identified by Theorem 4.2.2 with the dual $G^\wedge$ of G can be interpreted as the Gelfand space $\Delta(L^1(G, \omega_G))$ of $L^1(G, \omega_G)$. Since $\Delta(L^1(G, \omega_G))$ is a locally compact space with respect to the weak topology induced by the set

$$A(G^\wedge) := L^1(G, \omega_G)^\wedge,$$

$G^\wedge$ is also a locally compact *space*. This topology τ_G, sometimes named the **Gelfand topology** on $G^\wedge$, admits a neighborhood system of $\chi_0 \in G^\wedge$ of the form

$$V_{f_1,\dots,f_n;\varepsilon}(\chi_0) := \{\chi \in G^\wedge : |\widehat{f_i}(\chi) - \widehat{f_i}(\chi_0)| < \varepsilon \text{ for all } i = 1, \dots, n\},$$

where $f_1, \dots, f_n \in L^1(G, \omega_G)$ and $\varepsilon > 0$.

From the Gelfand theory we obtain the following essential

Properties 4.2.3 of $\mathcal{F}$.

4.2.3.1 $A(G^\wedge)$ *is a selfadjoint subalgebra of* $C^0(G^\wedge)$ *which separates* $G^\wedge$, *hence*

$$A(G^\wedge)^- = C^0(G^\wedge).$$

4.2.3.2 $\mathcal{F}$ *is a norm-decreasing involutive homomorphism of the group algebra* $L^1(G, \omega_G)$ *into* $C^0(G^\wedge)$.

Theorem 4.2.4 *The dual* $G^\wedge$ *of a locally compact group* G *is again a locally compact group.*

Proof. Since we already know that $G^\wedge$ is an Abelian group and a locally compact space with respect to the Gelfand topology τ_G it remains to be shown that the mapping

$$(\chi, \rho) \mapsto \chi - \rho$$

from $G^\wedge \times G^\wedge$ into $G^\wedge$ is continuous.

1. We prove that the mapping

$$(x, \chi) \mapsto \chi(x)$$

from $G \times G^\wedge$ into $\mathbb{C}$ is continuous.

First of all we note that for every $f \in L^1(G, \omega_G)$ and $x \in G$

$$\widehat{f_x}(\chi) = \hat{f}(\chi)\,\overline{\chi(x)}$$

holds whenever $\chi \in G^\wedge$. Thus it suffices to show that

$$(x, \chi) \mapsto \widehat{f_x}(\chi)$$

is continuous on $G \times G^\wedge$ for every $f \in L^1(G, \omega_G)$. So, take $x_0 \in G$, $\chi_0 \in G^\wedge$ and $\varepsilon > 0$. There are neighborhoods $V \in \mathfrak{V}_G(x_0)$ and $W \in \mathfrak{V}_{G^\wedge}(\chi_0)$ such that

$$\|f_x - f_{x_0}\|_1 < \varepsilon$$

as well as

$$\left|\widehat{f_{x_0}}(\chi) - \widehat{f_{x_0}}(\chi_0)\right| < \varepsilon$$

whenever $x \in V$, $\chi \in W$. This follows from Proposition 4.1.12 and from the continuity of f_{x_0} respectively. But since

$$\left|\widehat{f_x}(\chi) - \widehat{f_{x_0}}(\chi)\right| \leqslant \|f_x - f_{x_0}\|_1\,,$$

we obtain

$$\left|\widehat{f_x}(\chi) - \widehat{f_{x_0}}(\chi_0)\right| < 2\varepsilon$$

whenever $x \in V$ and $\chi \in W$, hence the assertion.

In the following two parts of the proof we shall employ the compact open topology τ_{co} in $G^\wedge$.

2. For compact $K \subset G$ and $r > 0$ the set

$$V_{K,r} := \{\chi \in G^\wedge : |\chi(x) - 1| < r \text{ for all } x \in K\}$$

is an open subset of $G^\wedge$.

In fact, we choose a compact subset K of G, an $r > 0$ and a $\chi_0 \in V_{K,r}$. Part 1. of this proof implies that for every $x_0 \in K$ there exist neighborhoods $V \in \mathfrak{V}_G(x_0)$ and $W \in \mathfrak{V}_{G^\wedge}(\chi_0)$ such that $|\chi(x) - 1| < r$ for all $x \in V$, $\chi \in W$. Since K is compact, finitely many of these sets V cover K, hence the intersection W_0 of the corresponding sets W is a subset of $V_{K,r}$. Since $W_0 \in \mathfrak{V}_{G^\wedge}(\chi_0)$, $V_{K,r}$ is open.

3. The family consisting of the sets $V_{K,r}$ and their translates generate the topology τ_{co} of $G^\wedge$.

In order to see this we pick a neighborhood V of $\chi_0 \in G^\wedge$ and show that $\chi_0 + V_{K,r} \subset V$ for some choice of $K \in \mathcal{K}(G)$ and $r > 0$. Without loss of generality let $\chi_0 = 0$. From the definition of the Gelfand topology τ_G in $G^\wedge$ we deduce the existence of functions $f_1, \ldots, f_n \in L^1(G, \omega_G)$ and of $\varepsilon > 0$ such that

$$\bigcap_{i=1}^n \{\chi \in G^\wedge : |\widehat{f_i}(\chi) - \widehat{f_i}(0)| < \varepsilon\} \subset V.$$

But now $C^c(G)$ is dense in $L^1(G, \omega_G)$, so we may assume that $f_1, \ldots, f_n$ vanish outside a compact subset K of G. With the choices

$$r < \frac{\varepsilon}{\max_{1 \leqslant i \leqslant n} \|f_i\|_1}$$

and $\chi \in V_{K,r}$ we get

$$\left|\widehat{f_i}(\chi) - \widehat{f_i}(0)\right| \leqslant \int_K |\overline{\chi(x)} - 1| \, |f_i(x)| \, \omega_G(\mathrm{d}x)$$

$$\leqslant r \|f_i\|_1 < \varepsilon,$$

hence that $V_{K,r} \subset V$.

4. The statement of the theorem now follows from the inclusion

$$(\chi + V_{K,\frac{r}{2}}) - (\rho + V_{K,\frac{r}{2}}) \subset \chi - \rho + V_{K,r}$$

valid for all $\chi, \rho \in G^\wedge$ and any $V_{K,r} \in \mathfrak{V}_{G^\wedge}(0)$. $\square$

Definition 4.2.5 From the proof of Theorem 4.2.4 we learn that on $G^\wedge \cong \Delta(L^1(G, \omega_G))$ the topologies τ_G and τ_{co} coincide. Therefore we may consider $G^\wedge$ as a locally compact Abelian *group* furnished with either of the topologies τ_G or τ_{co}. $G^\wedge$ will be called the **character group** or the dual (group) of G.

Since the norm of the spaces $L^p(G, \omega_G)$ for $p \in [1, \infty[$ is invariant (with respect to translation in G) we have

$$\|f * g\|_p \leqslant \|f\|_1 \|g\|_p$$

whenever $f \in L^1(G, \omega_G)$ and $g \in L^p(G, \omega_G)$, i.e. $L^1(G, \omega_G)$ operates linearly by means of $g \mapsto f * g$ on $L^p(G, \omega_G)$.

Let $\|f\|_T$ denote the norm of the corresponding operator defined on $L^2(G, \omega_G)$ by $f \in L^1(G, \omega_G)$. Then, clearly,

$$\|f\|_T \leqslant \|f\|_1 \, ,$$

and

$$\|f * g\|_2 \leqslant \|f\|_T \|g\|_2$$

for all $f \in L^1(G, \omega_G)$ and $g \in L^2(G, \omega_G)$.

The completion of $L^1(G, \omega_G)$ (or $C^c(G)$) with respect to the operator norm $\| \cdot \|_T$ is called the **extended group algebra** of G and will be denoted by $\Lambda(G)$.

Properties 4.2.6 of $\Lambda(G)$.

4.2.6.1 $\Lambda(G)$ *is a commutative Banach algebra containing* $L^1(G, \omega_G)$ *as a subalgebra.*

4.2.6.2 $\Lambda(G)$ *is an algebra of normal operators on the Hilbert space* $L^2(G, \omega_G)$ *and hence a commutative* C^**-algebra.*

4.2.6.3 $\Lambda(G)$ *admits a unit element if and only if* G *is discrete.*

4.2.6.4 *For any* $f \in \Lambda(G)$ *and* $\chi \in G^\wedge$ *we have*

$$\|\chi f\|_T = \|f\|_T \, .$$

For the Gelfand mapping $\mathcal{F} : f \mapsto \hat{f}$ on $\Lambda(G)$ we quote the following

Properties 4.2.7

4.2.7.1 $\mathcal{F}(\Lambda(G))$ *is a subalgebra of* $C^0(\Delta(\Lambda(G)))$.

4.2.7.2 $\mathcal{F}$ *is a norm and involution preserving isomorphism from* $\Lambda(G)$ *onto* $C^0(\Delta(\Lambda(G)))$.

4.2.7.3 $\Delta(\Lambda(G)) \cong G^\wedge$.

For the proof of 4.2.7.3 we take $f \in L^1(G, \omega_G)$ with $f \neq 0$. By Property 4.2.7.2 there exists $\tau' \in \Delta(\Lambda(G))$ such that $\tau'(f) \neq 0$ for all $f \in \Lambda(G)$. Now apply Theorem 4.2.2 in order to obtain $\chi_1 \in G^\wedge$ such that

$$\tau'(f) = \int_G f\overline{\chi}_1 \, \mathrm{d}\omega_G.$$

For any $\chi \in G^\wedge$ we put

$$\tau(f) := \tau'((\chi_1 - \chi)f) = \int f\overline{\chi} \, \mathrm{d}\omega_G$$

whenever $f \in L^1(G, \omega_G)$. Since

$$|\tau(f)| = |\tau'((\chi_1 - \chi)f)|$$
$$\leqslant \|(\chi_1 - \chi)f\|_T = \|f\|_T$$

for all $f \in L^1(G, \omega_G)$, τ can be uniquely extended to $\Lambda(G)$. Clearly, τ appears to be an element of $\Delta(\Lambda(G))$.

Let $\Lambda_\infty(G)$ and $\Lambda_2(G)$ denote the subsets of classes of $\|\cdot\|_T$-Cauchy sequences that contain at least one $\|\cdot\|$- or $\|\cdot\|_2$-Cauchy sequence respectively. In the sequel $\Lambda_\infty(G)$ and $\Lambda_2(G)$ will be viewed as subsets of $\Lambda(G)$. $\Lambda_2(G)$ will also be considered as a subset of $L^2(G, \omega_G)$.

Properties 4.2.8 of the spaces $\Lambda_\infty(G)$ and $\Lambda_2(G)$.

4.2.8.1 $\Lambda(G) * C^c(G) \subset \Lambda_2(G)$

4.2.8.2 $\Lambda_2(G) * \Lambda_2(G) \subset \Lambda_\infty(G)$

4.2.8.3 $\Lambda(G) * C^c(G) * C^c(G) \subset \Lambda_\infty(G)$

4.2.8.4 If $f \in \Lambda(G)$ such that $\hat{f}$ is real or $\hat{f} \geqslant 0$, then $f(0)$ is real or $f(0) \geqslant 0$ respectively.

The proof of Property 4.2.8.1 follows for $f \in \Lambda(G)$ of the form

$$f = \|\cdot\|_T - \lim_{n \to \infty} f_n$$

with a sequence $(f_n)_{n \geqslant 1}$ in $L^1(G, \omega_G)$ from the inequalities

$$\|f_n * g - f_m * g\|_2 \leqslant \|f_n - f_m\|_T \|g\|_2$$

and

$$\|f_n * g - f_m * g\|_T \leqslant \|f_n - f_m\|_T \|g\|_T$$

valid for $n, m \geqslant 1$.

Similarly, Property 4.2.8.2 is implied by the analogous estimates

$$\|f_n * g_n - f_m * g_m\| \leqslant \|(f_n - f_m) * g_n\| + \|f_m * (g_n - g_m)\|$$
$$\leqslant \|f_n - f_m\|_2 \|g_n\|_2 + \|f_m\|_2 \|g_n - g_m\|_2$$

and

$$\|f_n * g_n - f_m * g_m\|_T \leqslant \|f_n - f_m\|_T \|g_n\|_T + \|f_m\|_T \|g_n - g_m\|_T.$$

Property 4.2.8.3 follows from Properties 4.2.8.1 and 4.2.8.2 with the help of $C^c(G) \subset \Lambda_2(G)$.

Finally, as for Property 4.2.8.4, we assume $\hat{f} \geqslant 0$. Since by Property 4.2.7.2 the Gelfand isomorphism $f \mapsto \hat{f}$ from $\Lambda(G)$ onto $C^0(G^\wedge)$ is involution invariant, $\sqrt{\hat{f}} = \hat{g}$ yields a hermitian function g with $g * g = f$. This implies that

$$f(0) = g * g(0) = \int g(-y)g(y)\,\omega_G(\mathrm{d}y) = \int_G \overline{g^\sim(y)}g(y)\,\omega_G(\mathrm{d}y)$$
$$= \int \overline{g(y)}g(y)\,\omega_G(\mathrm{d}y) = \int_G |g(y)|^2\,\omega_G(\mathrm{d}y) \geqslant 0.$$

The case of $f \in \Lambda(G)$ such that $\hat{f}$ is real follows from the decomposition $f := g - h$, where $g, h \in \Lambda(G)$ with $\hat{g}, \hat{h} \geqslant 0$.

The aim of the following discussion is the proof of Pontryagin's fundamental theorem stating that any locally compact Abelian group G can be identified as a topological group with its double dual $G^{\wedge\wedge}$. On the way to this goal we shall establish two useful tools of harmonic analysis: the inversion formula for Fourier transforms and the Plancherel isomorphism.

Lemma 4.2.9 *Let $f \in \Lambda(G)$ such that $\hat{f} \in C^c_+(G^\wedge)$, and let $\varepsilon > 0$. There exist functions $f_1, f_2 \in \Lambda_\infty(G)$ satisfying*

(i) $\hat{f}_1, \hat{f}_2 \in C^c(G^\wedge)$,
(ii) $\hat{f}_1 \geqslant \hat{f} \geqslant \hat{f}_2$, *and*
(iii) $f_1(0) - f_2(0) < \varepsilon$.

Proof. (i). Let $C := \operatorname{supp}(\hat{f})$. There exists a neighborhood $U \in \mathfrak{V}_G(0)$ such that

$$|\hat{h}(\chi) - 1| < \varepsilon$$

for all $h \in C^c_+(G, U)$ with $\int_G h\,\mathrm{d}\omega_G = 1$ and for all $\chi \in C$. In order to see this one just chooses

$$U := \{x \in G : |\chi(x) - 1| < \varepsilon \text{ for all } \chi \in C\}.$$

Consequently there exists a $g_0 \in C_+^c(G)$ with $\hat{g}_0(\chi) \geqslant 1$ for all $\chi \in C$, and for each $\delta > 0$ there is a $g \in C_+^c(G)$ satisfying

$$1 + \delta \geqslant \hat{g}(\chi) \geqslant 1 - \delta$$

whenever $\chi \in C$. Now we define

$$f_1 := f * (g + \delta g_0)$$

and

$$f_2 := f * (g - \delta g_0)\,.$$

Since there exist $h, h_0 \in C^c(G)$ such that $g_0 = h_0 * h_0^{\sim}$ and $g = h * h^{\sim}$, we obtain that $f_1, f_2 \in \Lambda_\infty(G)$. And clearly, $\hat{f}_1, \hat{f}_2 \in C^c(G^\wedge)$.
(ii) follows from

$$\hat{f}_1 = \hat{f}\hat{g} + \delta\hat{f}\hat{g}_0 \geqslant \hat{f}(1 - \delta + \delta) = \hat{f}$$

and

$$\hat{f} = \hat{f}(1 + \delta - \delta) \geqslant \hat{f}\hat{g} - \delta\hat{f}\hat{g}_0 = \hat{f}_2\,.$$

(iii). Obviously,

$$f_1(0) - f_2(0) = 2\delta f * g_0(0)\,.$$

But

$$(f * g_0)^\wedge = \hat{f}\hat{g}_0 \geqslant 0$$

implies $f * g_0(0) \geqslant 0$. A proper choice of δ yields the assertion. □

Theorem 4.2.10 (Inversion)
Let $f \in \Lambda_\infty(G)$ with $\hat{f} \in C^c(G^\wedge)$. Then a Haar measure $\omega_{G^\wedge}$ on $G^\wedge$ can be chosen such that for all $x \in G$ the inversion formula

$$f(x) = \int_{G^\wedge} \hat{f}(\chi)\,\chi(x)\,\omega_{G^\wedge}(\mathrm{d}x)$$

holds.

Proof. For every $f \in \Lambda(G)$ with $\hat{f} \in C_+^c(G^\wedge)$ we introduce

$$F(\hat{f}) : = \sup\{g(0) : \hat{g} \leqslant \hat{f},\ f \in \Lambda_\infty(G)\}$$

$$= \inf\{h(0) : \hat{h} \geqslant \hat{f},\ h \in \Lambda_\infty(G)\}\,,$$

where the equalities defining the mapping $F : C_+^c(G^\wedge) \to \mathbb{C}$ are justified by Lemma 4.2.9. Obviously F is additive and positive homogeneous on $C_+^c(G^\wedge)$. Since $\mathrm{Re}\,C^c(G^\wedge)$ is a vector lattice, F can be extended to a linear functional on $\mathrm{Re}\,C^c(G^\wedge)$ and hence on $C^c(G^\wedge)$.

Obviously F is positive, since for every $f \in \Lambda_\infty(G)$ with $\hat{f} \geqslant 0$ we have $F(\hat{f}) = f(0) \geqslant 0$.

Moreover, $F \neq 0$. In fact, there exist an $f \in \Lambda(G)$, $f \neq 0$ with $\hat{f} \in C^c(G^\wedge)$ and a $g \in C^c(G)$ with $h := f * g \in \Lambda_2(G)$ and $h \neq 0$. But then $\ell := h * h^\sim \in \Lambda_\infty(G)$ and therefore

$$F(\hat{\ell}) = \ell(0) = h * h^\sim(0) = \|h\|_2^2 \neq 0.$$

In order to show the translation invariance of F it suffices to note that for every $\chi \in G^\wedge$

$$\hat{f}_\chi = (\chi f)^\wedge$$

and

$$\chi(0)\, f(0) = f(0).$$

But then there exists a Haar measure $\omega_{G^\wedge} \in M_+(G^\wedge)$ satisfying

$$F(\hat{f}) = \int_{G^\wedge} \hat{f}(\chi)\, \omega_{G^\wedge}(\mathrm{d}\chi)$$

for all $f \in \Lambda(G)$ with $\hat{f} \in C^c(G^\wedge)$.

For $f \in \Lambda_\infty(G)$ with $\hat{f} \in C^c(G^\wedge)$ we have

$$F(\hat{f}) = f(0),$$

hence

$$
\begin{aligned}
f(x) = f_{-x}(0) = F(\widehat{f_{-x}}) \\
= \int_{G^\wedge} \widehat{f_{-x}}(\chi)\, \omega_{G^\wedge}(\mathrm{d}\chi) \\
= \int_{G^\wedge} \hat{f}(\chi)\, \chi(x)\, \omega_{G^\wedge}(\mathrm{d}\chi),
\end{aligned}
$$

whenever $x \in G$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Theorem 4.2.11 (Plancherel)

(i) $\Lambda_2(G)$ is dense in $L^2(G, \omega_G)$.

(ii) The mapping $f \mapsto \hat{f}$ from $\Lambda_2(G)$ in $C^0(G^\wedge)$ is an isometry onto a dense subset of $L^2(G^\wedge, \omega_{G^\wedge})$ which can be extended uniquely to an isometry from $L^2(G, \omega_G)$ onto $L^2(G^\wedge, \omega_{G^\wedge})$.

(iii) For $f, g \in L^2(G, \omega_G)$ we have

$$\int f\overline{g}\, \mathrm{d}\omega_G = \int \hat{f}\overline{\hat{g}}\, \mathrm{d}\omega_{G^\wedge}.$$

Proof.   From Properties 4.2.7.2 and 4.2.7.3 we deduce the existence of a dense subset M of $\Lambda(G)$ with $M^\wedge = C^c(G^\wedge)$.

1. We show that the set

$$N := \{f * g : f \in M, \ g \in C^c(G)\}$$

is dense in $L^2(G, \omega_G)$. Since $N \subset \Lambda_2(G)$, we therefore obtain that $\Lambda_2(G)$ is dense in $L^2(G, \omega_G)$.

For the proof let $f \in L^2(G, \omega_G)$ and let $\varepsilon > 0$. First of all there is a $g \in C^c(G)$ with $\|f - g\| < \varepsilon$, and for this g there exists an $h \in C^c(G)$ with $\|g * h - g\|_2 < \varepsilon$. Next, for h there exists a $k \in M$ such that $\|h - k\|_T < \varepsilon$. Our assertion now follows from the inequality

$$\begin{aligned}
\|k * g - f\|_2 &\leqslant \|k * g - h * g\|_2 + \|h * g - g\|_2 + \|g - f\|_2 \\
&< \|k - h\|_T \|g\|_2 + \varepsilon + \varepsilon \\
&< \varepsilon(\varepsilon + \|f\|_2 + 2).
\end{aligned}$$

2. For every $f \in N$ we obtain that $\hat{f} \in C^c(G^\wedge)$ and $f * f^\sim \in \Lambda_\infty(G)$. Applying the inversion formula 4.2.10 the equalities

$$\begin{aligned}
\|f\|_2^2 &= \int_G f \overline{f} \, \mathrm{d}\omega_G = f * f^\sim(0) \\
&= \int_{G^\wedge} (f * f^\sim)^\wedge \, \mathrm{d}\omega_{G^\wedge} = \int_{G^\wedge} \hat{f}\overline{\hat{f}} \, \mathrm{d}\omega_{G^\wedge} = \|\hat{f}\|_2^2
\end{aligned}$$

show that $f \mapsto \hat{f}$ is an isometry on N. But this isometry extends to an isometry from $\Lambda_2(G)$ into $L^2(G^\wedge, \omega_{G^\wedge})$.

3. For the statements (i) and (ii) it remains to be shown that $\Lambda_2(G)^\wedge$ is dense in $L^2(G^\wedge, \omega_G)$.

In fact, let $\psi \in L^2(G^\wedge, \omega_G)$ with $\int_{G^\wedge} \varphi \hat{\psi} \, \mathrm{d}\omega = 0$ for all $\varphi \in \Lambda_2(G)^\wedge$. Then

$$\int_{G^\wedge} \check{\chi} \varphi \overline{\psi} \, \mathrm{d}\omega_{G^\wedge} = 0$$

whenever $\check{\chi} \in G^{\wedge\wedge}$, thus $(\varphi\overline{\psi})^\wedge = 0$ and hence $\varphi\overline{\psi} = 0$ $\omega_{G^\wedge}$-a.e. Since for each $\chi_0 \in G^\wedge$ there is a $\varphi \in \Lambda_2(G)^\wedge$ such that $\varphi \neq 0$ in some neighborhood of χ_0, $\psi = 0$ ω_G-a.e., the asserted density property has been shown.

4. For the proof of (iii) it suffices to refer to the well-known identity

$$4 f \overline{g} = |f + g|^2 - |f - g|^2 + \mathrm{i}|f + \mathrm{i}g|^2 - \mathrm{i}|f - \mathrm{i}g|^2$$

valid for all $f, g \in L^2(G, \omega_G)$. $\square$

Theorem 4.2.12 *The group algebra* $L^1(G, \omega_G)$ *is regular, i.e. to every proper closed subset C of $G^\wedge$ and to every $\chi \in C^c$ there exists an $f \in L^1(G, \omega_G)$ such that $\hat{f}(C) = 0$ and $\hat{f}(\chi) \neq 0$.*

Proof. Let C be a proper closed subset of $G^\wedge$. We shall show that there exists a function $\psi \in L^1(G, \omega_G)^\wedge$ satisfying $\psi(C) = 0$ and $\psi(\chi) \neq 0$ for $\chi \in U := \complement C$. Let $\chi = \chi_1 + \chi_2$ with $\chi_1, \chi_2 \in G^\wedge$. Then there are open sets U_1 and U_2 with $\chi_1 \in U_1$, $\chi_2 \in U_2$ and $U_1 + U_2 \subset U$. Now we choose functions $f, g \in L^2(G, \omega_G)$ with the properties $\hat{f}, \hat{g} \in C_+^c(G^\wedge)$, $\hat{f}(\chi_1) \neq 0$, $\hat{g}(\chi_2) \neq 0$, $\hat{f}(\complement U_1) = \hat{g}(\complement U_2) = 0$. Applying (iii) of Theorem 4.2.11 we have

$$(fg)^\wedge = \hat{f} * \hat{g},$$

hence $\psi := (fg)^\wedge$ satisfies the desired conditions: $\psi \in L^1(G, \omega_G)^\wedge$, $\psi(C) = \psi(\complement U) = 0$ and $\psi(\chi) \neq 0$. $\qquad\square$

Theorem 4.2.13 (Pontryagin)
Let G be a locally compact Abelian group with dual and double dual groups $G^\wedge$ and $G^{\wedge\wedge}$ respectively. For every $x \in G$ let $\Omega_x : G^\wedge \to \mathbb{C}$ be defined by

$$\Omega_x(\chi) := \chi(x)$$

for all $\chi \in G^\wedge$. Then the homomorphism $\Omega : G \to G^{\wedge\wedge}$ defined by

$$\Omega(x) := \Omega_x$$

for all $x \in G$ is a topological isomorphism.
    In short: $G^{\wedge\wedge} \cong G$.

Proof. 1. Ω as a mapping from (G, τ) into $(G^{\wedge\wedge}, \tau_{\mathrm{co}})$ is continuous.
    In fact, it is sufficient to show that Ω is continuous at $0 \in G$. Let C be a compact subset of $G^\wedge$ and $\varepsilon > 0$. Then the set

$$V_{C,\varepsilon} := \{\check{\chi} \in G^{\wedge\wedge} : |\check{\chi}(\chi) - 1| < \varepsilon \text{ for all } \chi \in C\}$$

is a neighborhood $W \in \mathfrak{V}_{G^{\wedge\wedge}}(0)$. By Part 1. of the proof of Theorem 4.2.4 the mapping $(x, \chi) \mapsto \chi(x)$ from $G \times G^\wedge$ into $\mathbb{C}$ is continuous. Since C is compact, the set

$$V := \{x \in G : |\chi(x) - 1| < \varepsilon \text{ for all } \chi \in C\}$$

belongs to $\mathfrak{V}_G(0)$, and $\Omega(V) \subset W$.

2. Ω is open.

It is to be shown that to each $U \in \mathfrak{V}_G(0)$ there exist a compact subset C of $G^\wedge$ and an $\varepsilon > 0$ such that

$$|\chi(x) - 1| < \varepsilon$$

for all $\chi \in C$ implies that $x \in U$.

In fact, we choose $g \in L^2(G, \omega_G)$ with $\|g\|_2 = 1$ such that

$$\|g - g_x\| < 1$$

implies $x \in U$. In order to see this, pick $V \in \mathfrak{V}_G(0)$ with $V^2 \subset U$. Moreover, take $g \in L_+^2(G, \omega_G)$ with $\mathrm{supp}(g) \subset V$ and $\|g\|_2 = 1$. If $x \notin V$ then $\mathrm{supp}(g) \cap \mathrm{supp}(g_x) = \emptyset$, hence

$$\|g - g_x\|_2 \geqslant \|g\|_2 = 1\,,$$

and we have a contradiction.

Next, let $f \in L_+^1(G, \omega_G)$ with $\|f\|_1 = 1$ satisfying the inequalities

$$\|f * g - g\|_2 < \frac{1}{3}$$

and

$$\|f * g_x - g_x\|_2 < \frac{1}{3}\,.$$

The triangle inequality immediately implies that

$$\|f * g - f * g_x\|_2 \leqslant \frac{1}{3}$$

yields $x \in U$. But since

$$\begin{aligned}
\|f * g - f * g_x\|_2 &= \|f * g - f_x * g\|_2 \\
&= \|(f - f_x) * g\|_2 \\
&\leqslant \|f - f_x\|_T\,,
\end{aligned}$$

we may continue the reduction procedure and obtain that

$$\|f - f_x\|_T \leqslant \frac{1}{3}$$

yields $x \in U$.

Moreover, we have that

$$\|f - f_x\|_T = \sup\{|\tau(f - f_x)| : \tau \in \Delta(\Lambda(G))\}$$

Thus, the inequality

$$|\tau(f - f_x)| < \frac{1}{3}$$

valid for all $\tau \in \Delta(\Lambda(G))$ implies $x \in U$. But

$$|\tau(f - f_x)| = |\tau(f)| \, |\chi_\tau(x) - 1| \,,$$

where χ_τ denotes the character associated with τ by Property 4.2.7.3. Now applying Property 4.2.7.1 we obtain that $\hat{f} := \tau(f) \in C^0(G^\wedge)$, thus there exists a compact subset C of $G^\wedge$ such that $\chi_\tau \notin C$ implies that

$$|\tau(f)| < \frac{1}{6} \,.$$

Now, choose $\varepsilon := \frac{1}{3\|f\|_T}$, and let $x \in G$ be such that

$$|\chi(x) - 1| < \varepsilon$$

for all $\chi \in C$. Then

$$|\tau(f - f_x)| < \frac{1}{3}$$

holds for all $\tau \in \Delta(\Lambda(G))$ and consequently $x \in U$.

3. $\Omega(G)$ is dense in $G^{\wedge\wedge}$.

Once we have shown this it will be clear that $\Omega(G)$ as the image of a locally compact and hence complete group G is itself complete and therefore closed in $G^{\wedge\wedge}$. The proof of the theorem will then be terminated.

Assume that $\Omega(G)$ is not dense in $G^{\wedge\wedge}$. Then by Theorem 4.2.12 there exists a $\varphi \in L^1(G^\wedge, \omega_{G^\wedge})$ with $\varphi \neq 0$ satisfying $\hat{\varphi}(\Omega(x)) = 0$ for all $x \in G$. Since $\varphi \neq 0$, there is a $g \in \Delta(G)$ such that

$$\int_{G^\wedge} \varphi(\chi)\hat{g}(\chi)\,\omega_{G^\wedge}(\mathrm{d}\chi) \neq 0 \,.$$

But $C^c(G)$ is dense in $\Delta(G)$, hence there also exists an $h \in C^c(G)$ such that

$$\int_{G^\wedge} \varphi(\chi)\hat{h}(\chi)\,\omega_{G^\wedge}(\mathrm{d}\chi) \neq 0 \,.$$

On the other hand, by assumption we have

$$\int_{G^\wedge} \varphi(\chi)\hat{h}(\chi)\,\omega_{G^\wedge}(\mathrm{d}\chi) = \int_{G^\wedge} \varphi(\chi)\left(\int_G h(x)\overline{\chi(x)}\,\omega_G(\mathrm{d}x)\right)\omega_{G^\wedge}(\mathrm{d}\chi)$$

$$= \int_G h(x)\left(\int_{G^\wedge} \varphi(\chi)\overline{\Omega(x)}(\chi)\omega_{G^\wedge}(\mathrm{d}\chi)\right)\omega_G(\mathrm{d}x)$$

$$= \int_G h(x)\hat{\varphi}(\Omega(x))\,\omega_G(\mathrm{d}x) = 0 \,,$$

which is the desired contradiction. $\qquad\qquad\square$

We draw a few consequences from Pontryagin's duality Theorem 4.2.13.

Theorem 4.2.14

(i) G *is compact if and only if* $G^\wedge$ *is discrete.*

(ii) G *is discrete if and only if* $G^\wedge$ *is compact.*

Proof.   An application of Theorem 4.2.13 reduces the proofs of (i) and (ii) to those of (i') and (ii') below.

(i') If G is compact, then $G^\wedge$ is discrete.

The only subgroup of $\mathbb{T}$ whose elements z satisfy $|z - 1| < \sqrt{3}$ is the trivial one. Therefore the set

$$V_{G,\sqrt{3}} := \{\chi \in G^\wedge : |\chi(x) - 1| < \sqrt{3} \ \text{ for all } \ x \in G\} = \{0\}$$

is a neighborhood in $\mathfrak{V}_{G^\wedge}(0)$. But this means that $G^\wedge$ is discrete.

(i'') If G is discrete, then $G^\wedge$ is compact.

It follows from the assumption that the function f_0 on G defined by $f_0(0) = 1$ and $f_0(x) = 0$ for all $x \in G$ with $x \neq 0$ is a unit of the group algebra $L^1(G, \omega_G)$. Consequently, by Preparation C.4.2 $\Delta(L^1(G, \omega_G))$ and by Theorem 4.2.2 also $G^\wedge$ is compact.   □

Definition 4.2.15 The **Fourier**(-Stieltjes) **transform** $\hat{\mu}$ of a measure $\mu \in M^b(G)$ is given by

$$\hat{\mu}(\chi) := \int_G \overline{\chi(x)} \, \mu(\mathrm{d}x)$$

for all $\chi \in G^\wedge$.

The **Fourier mapping**

$$\mathcal{F} = \mathcal{F}_G : M^b(G) \to C^b(G^\wedge)$$

is given by

$$\mathcal{F}(\mu) := \hat{\mu}$$

for all $\mu \in M^b(G)$.

Properties 4.2.16 of the Fourier mapping.

4.2.16.1 $\mathcal{F}$ *maps* $M^b(G)$ *into the space* $C^u(G^\wedge)$ *of uniformly continuous bounded functions on* $G^\wedge$.

4.2.16.2 $\mathcal{F}$ *is a norm-decreasing homomorphism of involutive algebras.*

4.2.16.3 $\mathcal{F}$ *is injective.*

By the isometric embedding of the group algebra $L^1(G, \omega_G)$ into the measure algebra $M^{\mathrm{b}}(G)$ analogous properties remain valid for the restriction of $\mathcal{F}$ to $L^1(G, \omega_G)$. In particular $M^{\mathrm{b}}(G)$ and $L^1(G, \omega_G)$ are semisimple Banach algebras.

Proof. We content ourselves with the proof of 4.2.16.3. By Pontryagin's Theorem 4.2.13 it suffices to show the injectivity of the **inverse Fourier mapping**

$$\check{\mathcal{F}} : M^{\mathrm{b}}(G^\wedge) \to C^{\mathrm{b}}(G)$$

defined by

$$\check{\mathcal{F}}(\mu) := \check{\mu}$$

for all $\mu \in M^{\mathrm{b}}(G^\wedge)$, where

$$\check{\mu}(x) := \overline{\mathcal{F}}_{G^\wedge}(\mu)(x) := \int_{G^\wedge} \chi(x)\, \mu(\mathrm{d}\chi)$$

whenever $x \in G$.

So, let $\mu \in M^{\mathrm{b}}(G^\wedge)$ such that $\check{\mu} = 0$. For every $f \in L^1(G, \omega_G)$ we have

$$\int_{G^\wedge} \hat{f}\, \mathrm{d}\mu = \int_{G^\wedge} \left(\int_G \overline{\chi(x)} f(x)\, \omega_G(\mathrm{d}x) \right) \mu(\mathrm{d}\chi)$$

$$= \int_G f(x)\, \omega_G(\mathrm{d}x) \int_{G^\wedge} \overline{\chi(x)}\, \mu(\mathrm{d}\chi) = 0\,.$$

Since $A(G^\wedge)$ is dense in $C^0(G^\wedge)$ by Property 4.2.3.1 we obtain that

$$\int_G g\, \mathrm{d}\mu = 0$$

for every $g \in C^0(G^\wedge)$ which implies that $\mu = 0$.   □

The following **functorial properties** of the duality of locally compact Abelian groups will be useful for its application to harmonic analysis.

Properties 4.2.17 *Let G and H be locally compact Abelian groups and let $\varphi : G \to H$ be a continuous homomorphism. For every $\chi \in H^\wedge$ let*

$$\varphi^\wedge(\chi) := \chi \circ \varphi \in G^\wedge\,.$$

Therefore, with the notation of Theorem 4.2.13,

$$\Omega_x(\varphi^\wedge(\chi)) = \Omega_{\varphi(x)}(\chi)\,,$$

whenever $x \in G$, $\chi \in H^\wedge$. Moreover, $\varphi^\wedge$ is a continuous homomorphism $H^\wedge \to G^\wedge$.

Indeed, the duality $\wedge$ *is a contravariant functor in the category of locally compact Abelian groups (together with continuous homomorphisms as morphisms).*

For any subset M of G let

$$M^{\perp} := \{\chi \in G^{\wedge} : \chi(x) = 1 \ \text{ for all } \ x \in M\}$$

be the **annihilator** or orthogonal complement of M.

Clearly $M^{\perp}$ is a closed subgroup of $G^{\wedge}$. From Pontryagin's theorem 4.2.13 we infer that $M^{\perp\perp} := (M^{\perp})^{\perp}$ is a closed subgroup of G. For $N \subset M$ one has $M^{\perp} \subset N^{\perp}$, and $M \subset M^{\perp\perp}$ which implies

$$M^{\perp} \supset (M^{\perp\perp})^{\perp} = (M^{\perp})^{\perp\perp} \supset M^{\perp},$$

hence $M^{\perp} = M^{\perp\perp\perp}$.

4.2.17.1 *The closed subgroup* $[M]^{-}$ *generated by a subset* M *of* G *coincides with* $M^{\perp\perp}$.

It is clear that $G_1 := [M]^{-} \subset M^{\perp\perp}$. For the remaining inclusion we consider the canonical projection π from G onto G/G_1 and take an element $x \notin G_1$. Since $G^{\wedge}$ separates G (by the Pontryagin theorem 4.2.13) there exists $\chi \in (G/G_1)^{\wedge}$ such that $\chi \circ \pi(x) \neq 1$. Hence $\chi \circ \pi \in G^{\wedge}$ with $\chi \circ \pi(y) = 1$ for all $y \in M$ but $\chi \circ \pi(x) \neq 1$, so $x \notin M^{\perp\perp}$.

As an immediate consequence of this property we note that

4.2.17.2 *the mapping*

$$H \to H^{\perp}$$

is a bijection from the class of closed subgroups of G *onto the class of closed subgroup of* $G^{\wedge}$.

4.2.17.3 $(G/H)^{\wedge}$ *and* $H^{\perp}$ *are (canonically) isomorphic locally compact Abelian groups.*

Again we define for every $\chi' \in (G/H)^{\wedge}$ the character $\chi \in G^{\wedge}$ by

$$\chi(y) := \chi'(y + H)$$

whenever $y \in G$. The mapping $\chi' \mapsto \chi$ from $(G/H)^{\wedge}$ into $G^{\wedge}$ serves as the desired topological isomorphism.

It follows

4.2.17.4 *that* $H^{\wedge}$ *is isomorphic to* $G^{\wedge}/H^{\perp}$.

One just has to observe the identifications

$$G^{\wedge}/H^{\perp} \cong (G^{\wedge}/H^{\perp})^{\wedge\wedge} \cong (H^{\perp\perp})^{\wedge} = H^{\wedge}.$$

4.2.17.5 H *is a compact subgroup of* G *if and only if* $H^{\perp}$ *is open in* $G^{\wedge}$.

This statement follows from Property 4.2.17.4 together with Theorem 4.2.14.

Examples 4.2.18 of locally compact Abelian groups and their duals.

4.2.18.1 $(\mathbb{R}^d)^\wedge \cong \mathbb{R}^d$, so $\mathbb{R}^d$ is **self dual** for any $d \geqslant 1$.

4.2.18.2 $(\mathbb{T}^d)^\wedge \cong \mathbb{Z}^d$ for $d \geqslant 1$.

4.2.18.3 $(\mathbb{Z}^d)^\wedge \cong \mathbb{T}^d$ for $d \geqslant 1$.

In order to establish these identifications the characters of the underlying groups have to be exhibited, and the Gelfand topology has to be recognized as the natural topology in each case.

In this context we mention the evident fact that

4.2.18.4 for locally compact Abelian groups $G_1, \ldots, G_n$ the identification

$$\left(\prod_{i=1}^n G_i \right)^\wedge \cong \prod_{i=1}^n G_i^\wedge$$

holds.

This property helps to see further special dualities once one accepts the validity of the following fundamental structure results.

Theorem 4.2.19 *Let G be a compactly generated locally compact Abelian group.*

Then

$$G \cong \mathbb{R}^d \times \mathbb{Z}^e \times K,$$

where $d, e \geqslant 0$ and K is a compact Abelian group.

From this theorem follows without difficulty

Theorem 4.2.20 (Pontryagin, van Kampen)
For every locally compact Abelian group G there exists an open subgroup G_1 of G of the form

$$G_1 \cong \mathbb{R}^d \times K,$$

where $d \geqslant 0$ and K denotes a compact Abelian group.
    If, in addition, G is connected, then

$$G \cong \mathbb{R}^d \times K,$$

where $d \geqslant 0$ and K is a connected compact Abelian group.

For a proof of these assertions the reader is referred for example to Guichardet [32].

4.3   Positive definite functions

The constituents of this title are important tools of the harmonic analysis of locally compact Abelian groups. For the first definition and its basic properties we note that a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ is said to be **positive hermitian** if for all $c_1, \ldots, c_n \in \mathbb{C}$

$$\sum_{i,j=1}^{n} a_{ij} c_i \overline{c}_j \geqslant 0 \,.$$

If $B = (b_{ij}) \in \mathbb{C}^{n \times n}$ is another positive hermitian matrix, then so is the product $AB = (d_{ij})$, where $d_{ij} := a_{ij} b_{ij}$ for all $i, j = 1, \ldots, n$.

Now let G be a locally compact group.

Definition 4.3.1 A complex-valued function φ on G is called **positive definite** if for all $n \geqslant 1$ and all $x_1, \ldots, x_n \in G$ the matrix $(\varphi(x_i - x_j)) \in \mathbb{C}^{n \times n}$ is positive hermitian.

The totality of positive definite functions on G will be abbreviated by $\mathrm{PD}(G)$. At a later stage we shall exclusively employ the set $\mathrm{CPD}(G) := \mathrm{PD}(G) \cap C(G)$.

Properties 4.3.2 of a function $\varphi \in \mathrm{PD}(G)$.

4.3.2.1 $\varphi^{\sim} = \varphi$ *and* $|\varphi| \leqslant \varphi(0)$, *in particular* φ *is bounded, and*

$$\sup_{x \in G} |\varphi(x)| = \varphi(0) \,.$$

4.3.2.2 *For any* $x, y \in G$ *we have*

$$|\varphi(x) - \varphi(y)|^2 \leqslant 2\varphi(0)(\varphi(0) - \mathrm{Re}\,\varphi(x - y)) \,.$$

4.3.2.3 *If* $\varphi(0) = 1$, *then*

$$|\varphi(x + y) - \varphi(x)\varphi(y)|^2 \leqslant (1 - |\varphi(x)|^2)(1 - |\varphi(y)|^2) \,.$$

4.3.2.4 *If* $\mathrm{Re}\,\varphi$ *is lower semicontinuous at* 0, *then* φ *is uniformly continuous.*

For proofs of these properties we consider specially chosen positive hermitian matrices.

Clearly, $\varphi(0) \geqslant 0$. Moreover, for every $x \in G$ the matrix

$$\begin{bmatrix} \varphi(0) & \varphi(-x) \\ \varphi(x) & \varphi(0) \end{bmatrix}$$

is positive hermitian. This fact takes care of Property 4.3.2.1.

Next, we consider the case $n = 3$, i.e. for any $x, y \in G$ we look at the matrix

$$\begin{bmatrix} \varphi(0) & \overline{\varphi(x)} & \overline{\varphi(y)} \\ \varphi(x) & \varphi(0) & \varphi(x-y) \\ \varphi(y) & \overline{\varphi(x-y)} & \varphi(0) \end{bmatrix}$$

under the assumption that $\varphi(x) \neq \varphi(y)$ we choose for $\lambda \in \mathbb{R}$ the complex numbers $c_1 := 1$, $c_2 := \lambda |\varphi(x) - \varphi(y)| (\varphi(x) - \varphi(y))^{-1}$, and $c_3 := -c_2$. The positive hermitian property of the above matrix yields the inequality

$$\varphi(0)(1 + 2\lambda^2) + 2\lambda |\varphi(x) - \varphi(y)| - 2\lambda^2 \operatorname{Re} \varphi(x-y) \geqslant 0$$

valid for all $\lambda \in \mathbb{R}$, and since the discriminant of the polynomial in λ occurring in this inequality is $\leqslant 0$, we obtain Property 4.3.2.2.

In order to show Property 4.3.2.3 we observe that a positive hermitian matrix of the form

$$\begin{bmatrix} 1 & z_1 & z_2 \\ \overline{z_1} & 1 & z_3 \\ \overline{z_2} & \overline{z_3} & 1 \end{bmatrix}$$

has a determinant $\geqslant 0$, hence

$$1 + z_1 \overline{z_2} z_3 + \overline{z_1} z_2 \overline{z_3} \geqslant |z_1|^2 + |z_2|^2 + |z_3|^2$$

or equivalently,

$$|z_3 - \overline{z_1} z_2|^2 \leqslant (1 - |z_1|^2)(1 - |z_2|^2).$$

Applying this inequality to the matrix in terms of φ and employing Property 4.3.2.1 implies the assertion.

Concerning Property 4.3.2.4 we note that from the assumption follows that $\operatorname{Re} \varphi$ is continuous at 0. This is a consequence of the equality

$$\{x \in G : \operatorname{Re} \varphi(x) \in]\varphi(0) - \varepsilon, \varphi(0) + \varepsilon[\}$$
$$= \{x \in G : \operatorname{Re} \varphi(x) > \varphi(0) - \varepsilon\}$$

together with $\operatorname{Re} \varphi \leqslant |\varphi| \leqslant \varphi(0)$ (Property 4.3.2.1). But then Property 4.3.2.2 provides the remaining argument.

4.3.2.5 *Let H be an open subgroup of G and let $\varphi \in \mathrm{CPD}(H)$. Then the function φ_0 defined by*

$$\varphi_0(x) := \begin{cases} \varphi(x) & \text{if } x \in H \\ 0 & \text{otherwise} \end{cases}$$

belongs to CPD(G).

For the proof of the positive-definiteness of φ_0 we pick a finite set F in G and for each $x \in F$, let $c_x \in \mathbb{C}$. The set F intersects only finitely many distinct cosets $w_1 + H, \ldots, w_n + H$ of H. With the notation $F_k := F \cap (w_k + H)$ for $k = 1, \ldots, n$ we then obtain

$$\sum_{x \in F} \sum_{y \in F} \varphi_0(x - y) c_x \overline{c_y} = \sum_{k=1}^{n} \sum_{x \in F_k} \sum_{y \in F_k} \varphi(x - y) c_x \overline{c_y} \,,$$

where

$$\sum_{x \in F_k} \sum_{y \in F_k} \varphi(x - y) c_x \overline{c_y}$$

$$= \sum_{x \in F_k} \sum_{y \in F_k} \varphi((x - w_k) - (y - w_k)) c_x \overline{c_y}$$

$$= \sum_{u \in F_k - w_k} \sum_{v \in F_k - w_k} \varphi(u - v) c_{u+w_k} \overline{c_{v+w_k}} \geqslant 0 \,.$$

Clearly, φ_0 extends φ continuously.

Properties 4.3.3 of the set PD(G).

4.3.3.1 *The set* PD(G) *is closed under formation of complex conjugates and real parts.*

4.3.3.2 *The constant function* $\geqslant 0$ *on* G *belongs to* PD(G).

4.3.3.3 PD(G) *is closed under formation of products.*

4.3.3.4 PD(G) *is a convex cone closed with respect to the topology* τ_{p}, CPD(G) *is a convex cone closed with respect to* τ_{co} *(in* $C(G)$).

Only Property 4.3.3.3 requires an argument, and this has been quoted at the beginning of the section.

Examples 4.3.4 Besides the constant function $\geqslant 0$

4.3.4.1 all characters χ of G are elements of PD(G).

This follows from the inequalities

$$\sum_{i,j=1}^{n} \chi(x_i - x_j) c_i \overline{c_j} = \left| \sum_{i=1}^{n} \chi(x_i) c_i \right|^2 \geqslant 0$$

valid for all $n \geqslant 1$, $x_1, \ldots, x_n \in G$ and $c_1, \ldots, c_n \in \mathbb{C}$.

On the other hand

4.3.4.2 any positive definite function $\varphi : G \to \mathbb{T}$ is a character of G, as is evident from Property 4.3.2.3.

4.3.4.3 The inverse Fourier transform of a measure $\mu \in M_+^{\mathrm{b}}(G^\wedge)$ belongs to $\mathrm{CPD}(G)$.

In fact, for all $n \geqslant 1$, $x_1, \ldots, x_n \in G$ and $c_1, \ldots, c_n \in \mathbb{C}$ we have as in 4.3.4.1

$$\sum_{i,j=1}^{n} \check{\mu}(x_i - x_j) c_i \bar{c}_j = \int \left| \sum_{i=1}^{n} \chi(x_i) c_i \right|^2 \mu(\mathrm{d}\chi) \geqslant 0,$$

hence $\check{\mu} \in \mathrm{PD}(G)$. The continuity of $\check{\mu}$ follows from Proposition 4.2.16.1.

The converse of this property is the statement of

Theorem 4.3.5 (Bochner)

For every $\varphi \in \mathrm{CPD}(G)$ there exists a unique measure $\beta := \beta_\varphi \in M_+^{\mathrm{b}}(G^\wedge)$ such that

$$\check{\mathcal{F}}(\beta) = \check{\beta} = \varphi.$$

*β is said to be the **Bochner measure** of φ. It satisfies $\|\beta\| = \varphi(0)$.*

Proof. 1. We first show that $\varphi \in \mathrm{CPD}(G)$ is of positive type in the sense that it satisfies

$$\int (f^* * f)\varphi \, \mathrm{d}\omega_G \geqslant 0$$

for all $f \in C^{\mathrm{c}}(G)$ and, since $C^{\mathrm{c}}(G)$ is dense in $L^1(G, \omega_G)$, also for all $f \in L^1(G, \omega_G)$. For a given $f \in C^{\mathrm{c}}(G)$ the function F on $G \times G$ defined by

$$F(x,y) := f(x) \, \overline{f(y)} \, \varphi(x - y)$$

for all $(x,y) \in G \times G$ belongs to $C^{\mathrm{c}}(G \times G)$, hence is uniformly continuous. For $K := \operatorname{supp} f$ we have $\operatorname{supp} F \subset K \times K$, and $K \times K$ can be covered by finitely many open sets $U \times U$ such that the variation of F on each of these sets is less than a prescribed $\varepsilon > 0$. By neglecting overlaps we therefore obtain a partition $\{E_1, \ldots, E_n\}$ of K and $x_l \in E_l$ $(l = 1, \ldots, n)$ such that $|F(x,y) - F(x_i, x_j)| < \varepsilon$ whenever $(x,y) \in E_i \times E_j$ $(i, j, \ldots, n)$. But then

$$\int (f^* * f)\varphi \, \mathrm{d}\omega_G = \iint F(x,y) \, \omega_G(\mathrm{d}x) \, \omega_G(\mathrm{d}y)$$

$$= \sum_{i,j=1}^{n} \int_{E_i} \int_{E_j} F(x,y) \, \omega_G(\mathrm{d}x) \, \omega_G(\mathrm{d}y)$$

$$= \sum_{i,j=1}^{n} F(x_i, x_j) \, \omega_G(E_i) \, \omega_G(E_j) + R$$

$$= \sum_{i,j=1}^{n} f(x_i) \, \omega_G(E_i) \, \overline{f(x_j) \, \omega_G(E_j)} \, \varphi(x_i - x_j) + R$$

where

$$|R| = \left| \sum_{i,j=1}^{n} \int_{E_i} \int_{E_j} (F(x,y) - F(x_i,x_j)) \, \omega_G(\mathrm{d}x) \, \omega_G(\mathrm{d}y) \right| < \varepsilon \, \omega_G(K)^2 \,.$$

Since by the positive definiteness of φ

$$\sum_{i,j=1}^{n} f(x_i) \, \omega_G(E_i) \, \overline{f(x_j) \, \omega_G(E_j)} \, \varphi(x_i - x_j) \geqslant 0$$

and ε was chosen arbitrarily, the assertion has been proved.

2. Next we construct a linear functional

$$\hat{f} \mapsto \int \varphi f \, \mathrm{d}\omega_G$$

on the Fourier algebra $A(G^{\wedge}) = L^1(G, \omega_G)^{\wedge}$. Without loss of generality we assume that $\varphi(0) = 1$. From the Schwarz inequality we deduce that the positive Hermitian (sesquilinear) form

$$(f,g) \mapsto [f,g]_{\varphi} := \int \varphi(f^* * g) \, \mathrm{d}\omega_G$$

on $L^1(G, \omega_G)$ satisfies

$$\left| [f,g]_{\varphi} \right|^2 \leqslant [f,f]_{\varphi} \, [g,g]_{\varphi}$$

for all $f, g \in L^1(G, \omega_G)$. Letting g run through an approximate identity $\{\psi_U : U \in \mathfrak{U}\}$ in $L^1(G, \omega_G)$ with

$$\psi_U^* * f \to f \quad (\text{in } L^1(G, \omega_G))$$

(Proposition 4.1.13) we obtain that

$$[\psi_U, f]_{\varphi} \to \int \varphi f \, \mathrm{d}\omega_G \quad \text{as } U \to \{0\}.$$

But along with $\{\psi_U : U \in \mathfrak{U}\}$ also $\{\psi_U^* * \psi_U : U \in \mathfrak{U}\}$ is an approximate identity in $L^1(G, \omega_G)$. In fact, if $\operatorname{supp} \psi_U \subset U$ then $\operatorname{supp}(\psi_U^* * \psi_U) \subset U - U$, and

$$\int \psi_U^* * \psi_U \, \mathrm{d}\omega_G = \left| \int \psi_U \, \mathrm{d}\omega_G \right|^2 = 1 \,.$$

It follows that

$$\left| \int \varphi f \, \mathrm{d}\omega_G \right|^2 \leqslant [f,f]_{\varphi}$$

for all $f \in L^1(G, \omega_G)$. For the function $h := f^* * f$ we have $h^* = h$. Applying the above inequality to the functions f, h, $h^{(2)} := h * h$, $h^{(3)} := h * h * h$, ... we obtain

$$\left| \int \varphi f \, \mathrm{d}\omega_G \right| \leqslant \left| \int \varphi h \, \mathrm{d}\omega_G \right|^{\frac{1}{2}}$$

$$\leqslant \left| \int \varphi h^{(2)} \, \mathrm{d}\omega_G \right|^{\frac{1}{4}}$$

$$\leqslant \ldots$$

$$\leqslant \left| \int \varphi h^{(2^n)} \, \mathrm{d}\omega_G \right|^{\frac{1}{2^{n+1}}}$$

$$\leqslant \|h^{(2^n)}\|_1^{\frac{1}{2^{n+1}}},$$

where $\|\varphi\|_\infty = \varphi(0) = 1$ has been applied. Now we infer from Theorem C.6 that

$$\lim_{n \to \infty} \|h^{(2^n)}\|_1^{\frac{1}{2^{n+1}}} = \|\hat{h}\|_\infty^{\frac{1}{2}} = \||\hat{f}|^2\|_\infty^{\frac{1}{2}} = \|\hat{f}\|_\infty .$$

Thus the mapping $f \mapsto \int \varphi f \, \mathrm{d}\omega_G$ induces a linear functional

$$\hat{f} \mapsto \int \varphi f \, \mathrm{d}\omega_G$$

on $A(G^\wedge)$.

3.   Since by Property 4.2.3.1 $A(G^\wedge)$ is dense in $C^0(G^\wedge)$ this linear functional extends to a linear functional F on $C^0(G^\wedge)$ with $\|F\| \leqslant 1$. We now apply the Riesz representation theorem 4.1.1 in order to obtain a measure $\nu \in M^b(G^\wedge)$ with $\|\nu\| \leqslant 1$ satisfying

$$F(f) = \int \hat{f} \, \mathrm{d}\nu = \iint f(x) \overline{\chi(x)} \, \nu(\mathrm{d}\chi) \, \omega_G(\mathrm{d}x)$$

for all $f \in L^1(G, \omega_G)$ or equivalently

$$\varphi(x) = \int \chi(x) \, \nu^*(\mathrm{d}\chi)$$

for all $x \in G$, hence $\varphi = \check{\beta}$ for $\beta := \nu^*$. From $1 = \varphi(0) = \beta(G^\wedge) \leqslant \|\beta\| \leqslant 1$ we conclude that $\beta(G^\wedge) = \|\beta\| \geqslant 0$, so that $\beta \in M_+^b(G^\wedge)$. The desired representation of φ has been established. $\square$

Remark 4.3.6 Replacing G by $G^\wedge$ and applying the Pontryagin theorem 4.2.13, Theorem 4.3.5 can be rephrased as follows: The Fourier mapping $\mathcal{F} := \mathcal{F}_G$ is a bijection of the cone $M_+^b(G)$ onto the cone $\mathrm{CPD}(G^\wedge)$. In particular, $\mathcal{F}$ maps the convex set $M^1(G)$ onto the convex set $\{\varphi \in \mathrm{CPD}(G^\wedge) : \varphi(0) = 1\}$.

We may now deepen our knowledge about this bijection by proving that

Theorem 4.3.7 $\mathcal{F}$ *is a homeomorphism of the cone* $(M_+^{\mathrm{b}}(G), \tau_{\mathrm{w}})$ *onto the cone* $(\mathrm{CPD}(G^\wedge), \tau_{\mathrm{co}})$.

Proof.   Let $(\mu_\alpha)_{\alpha \in A}$ be a net in $M_+^{\mathrm{b}}(G)$ such that

$$\mu_\alpha \xrightarrow{\tau_{\mathrm{w}}} \mu \in M_+^{\mathrm{b}}(G).$$

Then, clearly,

$$\hat{\mu}_\alpha(\chi) \to \hat{\mu}(\chi)$$

for all $\chi \in G^\wedge$.

1.   At first we show that for each $\varepsilon > 0$ there exists a $V \in \mathfrak{V}_{G^\wedge}(0)$ and there exists an $\alpha_0 \in A$ such that for all $\alpha \geqslant \alpha_0$ and for all $\chi_1, \chi_2 \in G^\wedge$ with $\chi_1 - \chi_2 \in V$ the inequality

$$|\hat{\mu}_\alpha(\chi_1) - \hat{\mu}_\alpha(\chi_2)| \leqslant \varepsilon$$

holds.

     In fact, let $\varepsilon > 0$ be given, choose $\delta > 0$ such that $\delta(3 + \|\mu\|) \leqslant \varepsilon$ and then pick $\varphi \in C_+^{\mathrm{c}}(G)$ with $0 \leqslant \varphi \leqslant 1$ and $\int (1 - \varphi)\, \mathrm{d}\mu < \delta$. Since $\mu_n \xrightarrow{\tau_{\mathrm{w}}} \mu$, there is an $\alpha_0 \in A$ such that the inequalities

$$\|\mu_\alpha\| < \|\mu\| + 1$$

and

$$\int (1 - \varphi)\, \mathrm{d}\mu_\alpha < \delta$$

are satisfied for all $\alpha \geqslant \alpha_0$. Now let V be a neighbourhood in $\mathfrak{V}_{G^\wedge}(0)$ of the form $V := V_{\mathrm{supp}\,\varphi, \delta}$. For $\alpha \in A$ and $\chi_1, \chi_2 \in G^\wedge$ with $\alpha \geqslant \alpha_0$, $\chi_1 - \chi_2 \in V$ we obtain

$$\begin{aligned}
|\hat{\mu}_\alpha(\chi_1) - \hat{\mu}_\alpha(\chi_2)| &\leqslant \int |\chi_1(x) - \chi_2(x)|\, \mu_\alpha(\mathrm{d}x) \\
&\leqslant \int |1 - (\chi_1 - \chi_2)(x)|\, \varphi(x)\, \mu_\alpha(\mathrm{d}x) \\
&\quad + \int |1 - (\chi_1 - \chi_2)(x)|\, (1 - \varphi(x))\, \mu_\alpha(\mathrm{d}x) \\
&\leqslant \delta \int \varphi(x)\, \mu_\alpha(\mathrm{d}x) + 2 \int |1 - \varphi(x)|\, \mu_\alpha(\mathrm{d}x) \\
&\leqslant \delta(\|\mu\| + 1) + 2\delta \leqslant \varepsilon.
\end{aligned}$$

2. Next we show that

$$\hat{\mu}_\alpha \xrightarrow{\ \tau_{\text{co}}\ } \hat{\mu}\,.$$

Let K be a compact subset of $G^\wedge$, and let $\varepsilon > 0$. We choose α_0 and V as above, and by taking the limits along α it follows that

$$|\hat{\mu}(\chi_1) - \hat{\mu}(\chi_2)| \leqslant \varepsilon$$

whenever $\chi_1, \chi_2 \in G^\wedge$ satisfy $\chi_1 - \chi_2 \in V$. Since K is compact, there exist $\chi_1, \ldots, \chi_n \in K$ such that $K \subset \bigcup_{i=1}^n (\chi_i + V)$, hence there exist $\alpha_1, \ldots, \alpha_n \in A$ with

$$|\hat{\mu}_\alpha(\chi_i) - \hat{\mu}(\chi_i)| \leqslant \varepsilon$$

for all $\alpha \geqslant \alpha_i$ $(i = 1, \ldots, n)$. Let $\alpha^* \in A$ be chosen such that $\alpha^* > \alpha_i$ for $i = 0, 1, \ldots, n$. Then, for $\chi \in \chi_i + V$ and $\alpha \geqslant \alpha^*$ we obtain the estimate

$$|\hat{\mu}_\alpha(\chi) - \hat{\mu}(\chi)| \leqslant |\hat{\mu}_\alpha(\chi) - \hat{\mu}_\alpha(\chi_i)| + |\hat{\mu}_\alpha(\chi_i) - \hat{\mu}(\chi_i)| + |\hat{\mu}(\chi_i) - \hat{\mu}(\chi)|$$
$$\leqslant 3\,\varepsilon$$

and hence that

$$\sup_{\chi \in K} |\hat{\mu}_\alpha(\chi) - \hat{\mu}(\chi)| \leqslant 3\,\varepsilon$$

which is the desired statement.

3. We now suppose that

$$\hat{\mu}_\alpha \xrightarrow{\ \tau_{\text{co}}\ } \hat{\mu}$$

and show that

$$\mu_\alpha \xrightarrow{\ \tau_{\text{w}}\ } \mu\,.$$

From the hypothesis follows that

$$\lim_\alpha \|\mu_\alpha\| = \lim_\alpha \hat{\mu}_\alpha(0) = \hat{\mu}(0) = \|\mu\|\,.$$

Therefore, as a consequence of Proposition 4.1.3 it suffices to verify the limit relationship

$$\mu_\alpha \xrightarrow{\ \tau_{\text{v}}\ } \mu\,.$$

For $\varphi \in C_+^{\text{c}}(G)$ and $\varepsilon > 0$ we choose $f \in C^{\text{c}}(G^\wedge)$ such that

$$\|\varphi - \mathcal{F}_{G^\wedge} f\| < \varepsilon\,.$$

But then
$$\left| \int \varphi \, d\mu_\alpha - \int \varphi \, d\mu \right| \leqslant \left| \int (\varphi - \mathcal{F}_{G^\wedge} f) \, d\mu_\alpha \right| + \left| \int (\mathcal{F}_{G^\wedge} f - \varphi) d\mu \right|$$
$$+ \left| \int \mathcal{F}_{G^\wedge} f \, d\mu_\alpha - \int \mathcal{F}_{G^\wedge} f \, d\mu \right|$$
$$\leqslant \varepsilon(\|\mu_\alpha\| + \|\mu\|)$$
$$+ \int |\hat{\mu}_\alpha(\chi) - \hat{\mu}(\chi)| \, |f(\chi)| \, \omega_{G^\wedge}(d\chi) \,,$$

hence by assumption
$$\limsup_\alpha \left| \int \varphi \, d\mu_\alpha - \int \varphi \, d\mu \right| \leqslant 2\varepsilon \|\mu\|$$

which implies the assertion. $\square$

Theorem 4.3.8 (Sequential continuity of the Fourier transform)
Let $(\mu_n)_{n \geqslant 1}$ *be a sequence of measures in* $M_+^b(G)$, *and let* φ *be a complex-valued function on* $G^\wedge$ *which is continuous at* $0 \in G^\wedge$ *such that*
$$\hat{\mu}_n(\chi) \to \varphi(\chi)$$
for all $\chi \in G^\wedge$. *Then there exists a measure* $\mu \in M_+^b(G)$ *with* $\hat{\mu} = \varphi$ *such that*
$$\mu_n \xrightarrow{\tau_w} \mu \,.$$

Proof.   Clearly, $\hat{\mu}_n \in \mathrm{CPD}(G^\wedge)$ for all $n \geqslant 1$. From Property 4.3.3.4 we infer that
$$\varphi = \lim_{n \to \infty} \hat{\mu}_n \in \mathrm{PD}(G^\wedge) \,,$$
and Property 4.3.2.4 yields that $\varphi \in C(G^\wedge)$, hence $\varphi \in \mathrm{CPD}(G^\wedge)$. Now we apply the Bochner theorem 4.3.5 and obtain a measure $\mu \in M_+^b(G)$ satisfying $\hat{\mu} = \varphi$. It remains to be shown that for all $\psi \in C_+^c(G^\wedge)$
$$\lim_{n \to \infty} \int \psi \, d\mu_n = \int \psi \, d\mu \,.$$
As in part 3. of the proof of Theorem 4.3.7 with
$$\|\psi - \mathcal{F}_{G^\wedge} f\| < \varepsilon$$
for $\varepsilon > 0$ and $f \in C^c(G^\wedge)$ we establish the inequality
$$\left| \int \psi \, d\mu_n - \int \psi \, d\mu \right|$$
$$\leqslant \varepsilon(\|\mu_n\| + \|\mu\|) + \int_G |\hat{\mu}_n(\chi) - \hat{\mu}(\chi)| \, |f(\chi)| \, \omega_{G^\wedge}(d\chi) \,.$$
But the dominated convergence theorem implies
$$\lim_{n \to \infty} \int |\hat{\mu}_n(\chi) - \hat{\mu}(\chi)| \, |f(\chi)| \, \omega_{G^\wedge}(d\chi) = 0 \,,$$
hence the assertion. $\square$

4.4 Positive definite measures

We proceed to the study of Fourier transforms of not necessarily bounded positive definite measures on a locally compact Abelian group G. Our next aim will be to prove an analog of Theorem 4.3.7 for nonnegative positive definite measures on G.

Some measure-theoretical supplements will facilitate the comprehension.

Definition 4.4.1 A measure $\mu \in M(G)$ is said to be **shift bounded** if
$$\mu * C^c(G) \subset C^b(G).$$
It is easily seen that μ is shift bounded if and only if the set $\{T_a(\mu) : a \in G\}$ of translates of μ is τ_v-bounded.

Properties 4.4.2

4.4.2.1 *Along with* μ *also* μ^* *and* $\mu^\sim$ *are shift bounded measures.*

4.4.2.2 *If* μ *is shift bounded, also* $|\mu|$ *is shift bounded.*

4.4.2.3 *Any pair* (μ, ν) *consisting of a shift bounded measure* μ *and a bounded measure* ν *is convolvable.*

4.4.2.4 *A measure* $\mu \in M_+(G)$ *is shift bounded if and only if for each set* $K \in \mathcal{K}(G)$ *the function*
$$x \mapsto \mu(K + x)$$
is bounded.

Only Properties 4.4.2.2 and 4.4.2.3 deserve an argument. Concerning Property 4.4.2.2 we look at the inductive limit representation
$$C^c(G) = \varinjlim_{K \in \mathcal{K}(G)} C^c(G, K)$$
in the sense of Appendix B.4. Suppose that $f \in C_+^c(G, K)$. Then for any $g \in C^c(G)$ with $|g| \leqslant f$ we have
$$\|\mu * g\|_\infty \leqslant C_K \|g\|_\infty \leqslant C_K \|f\|_\infty \,,$$
where C_K is a constant $\geqslant 0$. In particular we obtain for all $x \in G$ that
$$|\mu| * f(x) = \int (f^*)_x \, \mathrm{d}|\mu|$$
$$= \sup_{|g| \leqslant f} \left| \int (g^*)_x \, \mathrm{d}\mu \right|$$
$$= \sup_{|g| \leqslant f} |\mu * g(x)|$$
$$\leqslant C_K \|f\|_\infty$$

which says that $|\mu|$ is shift bounded.

As for Property 4.4.2.3 we just note that for each $f \in C_+^c(G)$ we have

$$\int \left(\int f(x+y)\,|\mu|(\mathrm{d}x) \right) |\nu|(\mathrm{d}y) = \int (|\mu^*| * f)(y)\,|\nu|(\mathrm{d}y) < \infty \,,$$

since $|\mu^*| * f$ is a bounded function on G.

Definition 4.4.3 A measure $\mu \in M(G)$ is said to **vanish at infinity** if

$$\mu * C^c(G) \subset C^0(G) \,.$$

With the obvious notation $M^{\mathrm{sb}}(G)$ and $M^\infty(G)$ for the measures defined in 4.4.1 and 4.4.3 respectively we note that

$$M^{\mathrm{b}}(G) \subset M^\infty(G) \subset M^{\mathrm{sb}}(G) \,.$$

Examples 4.4.4

4.4.4.1 The Haar measure ω_G of G belongs to $M^{\mathrm{sb}}(G)$, and

4.4.4.2 $\omega_G \in M^\infty(G)$ if and only if G is compact.

Properties 4.4.5

4.4.5.1 *For* $\mu \in M^{\mathrm{sb}}(G)$ *the linear mapping* $f \mapsto \mu * f$ *from* $C^c(G)$ *into* $C^{\mathrm{b}}(G)$ *is continuous.*

4.4.5.2 *Moreover*

$$\mu * C^c(G) \subset C^{\mathrm{u}}(G)$$

whenever $\mu \in M^{\mathrm{sb}}(G)$.

While Property 4.4.5.1 is an easy consequence of Appendix B.6 (closed graph theorem), Property 4.4.5.2 requires a proof. Given $\varepsilon > 0$ and a compact symmetric neighbourhood $V_0 \in \mathfrak{V}_G(0)$. Then Property 4.4.2.4 implies that

$$\alpha := \sup_{x \in G} |\mu|\big(V_0 - \mathrm{supp}(f) + x\big) < \infty \,.$$

Since $f \in C^{\mathrm{u}}(G)$, there exists a $V \in \mathfrak{V}_G(0)$ with $V \subset V_0$ such that

$$|f(x) - f(y)| \leqslant \frac{\varepsilon}{\alpha}$$

for all $x, y \in G$ with $x - y \in V$. But for such x, y it follows that

$$|\mu * f(x) - \mu * f(y)| \leqslant \int |f(x - z) - f(y - z)|\,|\mu|(\mathrm{d}z)$$

$$\leqslant \frac{\varepsilon}{\alpha}|\mu|\big(V_0 - \mathrm{supp}(f) + x\big) \leqslant \varepsilon \,.$$

As a motivation for the notion of positive definite measures on G we note that a function $\varphi \in C(G)$ is positive definite if and only if it is of **positive type** in the sense of the inequality

$$\int \varphi(f * f^{\sim}) \, d\omega_G \geqslant 0$$

valid for all $f \in C^c(G)$. (See part 1. of the proof of Theorem 4.3.5.)

Definition 4.4.6 A measure $\mu \in M(G)$ is called **positive definite** if for all $f \in C^c(G)$ one has

$$\int f * f^{\sim} \, d\mu \geqslant 0 \, .$$

The set of positive definite measures on G will be denoted by $M_p(G)$.

Obviously $M_p(G)$ is a τ_v-closed cone in $M(G)$ which is stable under formation of reflections and complex conjugates.

As first

Examples 4.4.7 of positive definite measures we mention ε_0 and ω_G.

Facts 4.4.8

4.4.8.1 *For a function $\varphi \in C(G)$ such that $\varphi \cdot \omega_G \in M_p(G)$ it is necessary and sufficient that $\varphi \in \mathrm{PD}(G)$.*

4.4.8.2 *If for $\mu \in M(G)$ the pair $(\mu, \mu^{\sim})$ is convolvable, then $\mu * \mu^{\sim} \in M_p(G)$.*

4.4.8.3 *If $\varphi \in \mathrm{CPD}(G)$ with Bochner measure $\beta \in M_+^b(G^{\wedge})$, then for any $f \in C^c(G)$ the function $\varphi * f * f^{\sim}$ belongs to $\mathrm{CPD}(G)$ and has Bochner measure $|\hat{f}|^2 \cdot \beta$.*

In order to see this we show that the inverse Fourier transform of the measure $|\hat{f}|^2 \cdot \beta \in M_+^b(G^{\wedge})$ is $\varphi * f * f^{\sim}$, and this in turn follows from the subsequent computation valid for all $x \in G$:

$$\overline{\mathcal{F}}_{G^{\wedge}}(|\hat{f}|^2 \cdot \beta) = \int \left(\chi(x) \int \overline{\chi(y)} f * f^{\sim}(y) \, \omega_G(dy) \right) \beta(d\chi)$$

$$= \int \left(\int \chi(y) f * f^{\sim}(x - y) \, \omega_G(dy) \right) \beta(d\chi)$$

$$= \int \varphi(y) f * f^{\sim}(x - y) \, \omega_G(dy)$$

$$= \varphi * f * f^{\sim}(x) \, .$$

With this statement as a motivation we proceed by looking at the positive definiteness of functions $\mu * f * f^\sim$ with measures μ instead of functions φ.

Proposition 4.4.9 *Let $\mu \in M(G)$.*

(i) $\mu \in M_{\mathrm{p}}(G)$ *if and only if* $\mu * f * f^\sim \in \mathrm{CPD}(G)$ *for all* $f \in C^{\mathrm{c}}(G)$.
(ii) *If* $\mu \in M_{\mathrm{p},+}(G) := M_{\mathrm{p}}(G) \cap M_+(G)$, *then* $\mu \in M^{\mathrm{sb}}(G)$.

Proof.   (i). Let $\mu \in M_{\mathrm{p}}(G)$ and let $f, g \in C^{\mathrm{c}}(G)$. Then $\mu * f * f^\sim \in C(G)$, and

$$\int \mu * f * f^\sim(g * g^\sim)\,\mathrm{d}\omega_G = \int (f^* * g) * (f^* * g)^\sim \,\mathrm{d}\mu \geqslant 0\,.$$

The statement preceding Definition 4.4.6 implies that $\mu * f * f^\sim \in \mathrm{CPD}(G)$.

Conversely, if $\mu * f * f^\sim \in \mathrm{CPD}(G)$ for all $f \in C^{\mathrm{c}}(G)$, then $\mu * f * f^\sim(0) \geqslant 0$ which says that

$$\int f^* * (f^*)^\sim \,\mathrm{d}\mu \geqslant 0$$

for all $f \in C^{\mathrm{c}}(G)$, hence that $\mu \in M_{\mathrm{p}}(G)$.

(ii). Let $\mu \in M_{\mathrm{p},+}(G)$ and $f \in C_+^{\mathrm{c}}(G)$. There exists a function $g \in C_+^{\mathrm{c}}(G)$ such that $f \leqslant g * g^\sim$. It follows that

$$\mu * f \leqslant \mu * g * g^\sim\,,$$

where $\mu * g * g^\sim \in \mathrm{CPD}(G)$ (by (i)) and hence bounded. Consequently $\mu * f \in C^{\mathrm{b}}(G)$, i.e. $\mu \in M^{\mathrm{sb}}(G)$. $\square$

Theorem 4.4.10 (Existence of generalized Bochner measure)
Let $\mu \in M_{\mathrm{p}}(G)$.   There exists a unique measure $\beta \in M(G^\wedge)$ such that for all $f \in C^{\mathrm{c}}(G)$ the following conditions hold:

(i) $\int |\hat{f}|^2 \,\mathrm{d}|\beta| < \infty$.
(ii) $\mu * f * f^\sim(x) = \int \chi(x)|\hat{f}(\chi)|^2 \,\beta(\mathrm{d}\chi)$ *whenever* $x \in G$.

β *is called the **generalized Bochner measure** associated with μ.*

Proof.   We first note that by Proposition 4.4.9 (i) $\mu * f * f^\sim \in \mathrm{CPD}(G)$ for all $f \in C^{\mathrm{c}}(G)$. Now Theorem 4.3.5 provides us with a Bochner measure $\beta_f \in M_+^{\mathrm{b}}(G^\wedge)$ satisfying

$$\mu * f * f^\sim = \check{\beta}_f\,.$$

Since for $f, g \in C^{\mathrm{c}}(G)$

$$(|\hat{f}|^2 \beta_g)^\vee = \mu * f * f^\sim * g * g^\sim = (|\hat{g}|^2 \beta_f)^\vee\,,$$

the uniqueness of the Bochner measure yields

$$|\hat{f}|^2 \beta_g = |\hat{g}|^2 \beta_f . \tag{1}$$

Given $\beta \in M(G^\wedge)$ with the properties (i) and (ii) of the theorem one has

$$|\hat{f}|^2 \cdot \beta = \beta_f \tag{2}$$

whenever $f \in C^c(G)$. From this we see that the evaluation $\beta(h)$ of β at h is uniquely determined for every $h \in C^c(G^\wedge)$. One only chooses $g \in C^c(G)$ with $\hat{g} \neq 0$ on $\operatorname{supp} h$ (by applying Property 4.2.3.1) and observes that

$$\beta(h) = \int_{G^\wedge} \frac{h}{|\hat{g}|^2} \, \mathrm{d}\beta_g,$$

where $\frac{h}{|\hat{g}|^2}$ denotes the function in $C^c(G^\wedge)$ given as

$$\chi \mapsto \begin{cases} \frac{h(\chi)}{|\hat{g}(\chi)|^2} & \text{if } \hat{g}(\chi) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

So, given $g \in C^c(G)$ with $\hat{g} \neq 0$ on $\operatorname{supp} h$, $\beta(h)$ can in fact be defined by the integral representation above, since by (1) this is easily seen to be independent of the choice of g. Clearly, $h \mapsto \beta(h)$ is a positive linear functional on $C^c(G^\wedge)$, hence by the Riesz representation theorem 4.1.1 $\beta \in M_+(G^\wedge)$. It remains to show that β satisfies the conditions (i) and (ii) of the theorem. For this it suffices to establish the equality (2).

Indeed let $h \in C^c(G^\wedge)$, choose again $g \in C^c(G)$ with $\hat{g} \neq 0$ on $\operatorname{supp}(h)$ and apply again (1) together with the definition of β. Then the equalities

$$\int_{G^\wedge} |\hat{f}|^2 h \, \mathrm{d}\beta = \int_{G^\wedge} \frac{|\hat{f}|^2 h}{|\hat{g}|^2} \, \mathrm{d}\beta_g$$
$$= \int_{G^\wedge} \frac{h}{|\hat{g}|^2} |\hat{g}|^2 \, \mathrm{d}\beta_f$$
$$= \int_{G^\wedge} h \, \mathrm{d}\beta_f$$

imply the assertion. $\qquad\square$

Theorem 4.4.11 *For any measure $\mu \in M(G)$ the following statements are equivalent:*

(i) $\mu \in M_{\mathrm{p}}(G)$.

(ii) *There exists a measure* $\sigma \in M_+(G^\wedge)$ *such that*

$$\int f * f^\sim \, \mathrm{d}\mu = \int |\overline{\mathcal{F}}_G f|^2 \, \mathrm{d}\sigma \qquad (3)$$

whenever $f \in C^c(G)$.

If (ii) *is fulfilled, then* σ *is the generalized Bochner measure* β_μ *associated with* μ.

Proof. (ii) $\Rightarrow$ (i). Let μ be a measure in $M(G)$ for which there exists $\sigma \in M_+(G^\wedge)$ satisfying the equality (3) valid for all $f \in C^c(G)$. Then obviously μ belongs to $M_\mathrm{p}(G)$. We now fix $f \in C^c(G)$ and $x \in G$. With f replaced by f^* (ii) yields

$$\int |\hat{f}|^2 \, \mathrm{d}\sigma = \int f^* * \overline{f} \, \mathrm{d}\mu < \infty \, ,$$

i.e. condition (i) of Theorem 4.4.10. Moreover, polarization of the equality in (ii) implies

$$\int f * g^\sim \, \mathrm{d}\mu = \int \overline{\mathcal{F}}_G f \, \overline{\overline{\mathcal{F}}_G g} \, \mathrm{d}\sigma$$

whenever $g \in C^c(G)$. Replacing f by $\overline{f}_x$ and g by $\overline{f}$ shows that condition (ii) of Theorem 4.4.10 is fulfilled and therefore $\sigma = \beta$.

(i) $\Rightarrow$ (ii). If conversely $\mu \in M_\mathrm{p}(G)$ then the generalized Bochner measure β associated with μ (by Theorem 4.4.10) satisfies the equality (3) for each $f \in C^c(G)$. One needs only replace f by f^* and specialize condition (ii) of Theorem 4.4.10 to $x = 0$. $\square$

Corollary 4.4.12 *The mapping* $\mu \mapsto \beta_\mu$ *from* $M_\mathrm{p}(G)$ *into* $M_+(G^\wedge)$ *established in the theorem (and envisaged to serve as a generalization of the Fourier mapping) is injective.*

Proof. Suppose that measures $\mu, \nu \in M_\mathrm{p}(G)$ admit the same generalized Bochner measure β. By the theorem we have that

$$\int f * f^\sim \, \mathrm{d}\mu = \int f * f^\sim \, \mathrm{d}\nu$$

for all $f \in C^c(G)$, and by polarization we immediately obtain

$$\int f * g^\sim \, \mathrm{d}\mu = \int f * g^\sim \, \mathrm{d}\nu$$

whenever $f, g \in C^c(G)$. Letting g run through an approximate identity in $C^c(G)$ (see Proposition 4.1.13) we achieve that

$$\int f \, \mathrm{d}\mu = \int f \, \mathrm{d}\nu$$

holds for all $f \in C^c(G)$, i.e. $\mu = \nu$. $\square$

Consequences 4.4.13 of the preceding discussion.

4.4.13.1 If $\mu \in M_p(G)$ then $\beta_\mu \in M^{sb}(G^\wedge)$. If, in addition, $\mu \ll \omega_G$ then $\beta_\mu \in M^\infty(G^\wedge)$.

We show the first statement. Clearly,

$$\chi(f * f^\sim) = (\chi f) * (\chi f)^\sim$$

for all $\chi \in G^\wedge$, $f \in C^c(G)$. Replacing f by $\overline{\chi} f$ in (3) this implies

$$\int \overline{\chi(x)} f * f^\sim(x)\, \mu(\mathrm{d}x) = \int |\hat{f}(\chi - \varrho)|^2 \beta_\mu(\mathrm{d}\varrho)$$
$$= \beta_\mu * |\hat{f}|^2(\chi).$$

We observe that

$$\beta_\mu * |\hat{f}|^2 = ((f * f^\sim) \cdot \mu)^\wedge$$

is a bounded function, since $(f * f^\sim) \cdot \mu \in M^b(G)$. But for every compact $C \subset G^\wedge$ there exists a function $f \in C_+^c(G)$ satisfying $\mathbb{1}_C \leqslant \hat{f}$ (which is a consequence of Property 4.2.3.1), or for every function $\psi \in C_+^c(G^\wedge)$ there exists a function $f \in C^c(G)$ with $\psi \leqslant |\hat{f}|^2$, hence $\beta_\mu * \psi \in C^b(G^\wedge)$ for all $\psi \in C_+^c(G^\wedge)$.

4.4.13.2 Let $\mu \in M_p(G)$ with generalized Bochner measure β_μ, and let $\varphi \in \mathrm{CPD}(G)$ with Bochner measure β_φ. Then $\varphi \cdot \mu \in M_p(G)$, $(\beta_\mu, \beta_\varphi)$ is a convolvable pair, and

$$\beta_\mu * \beta_\varphi = \beta_{\varphi \cdot \mu}.$$

From 4.4.13.1 we infer that $\beta_\mu \in M^{sb}(G^\wedge)$. For every $f \in C^c(G)$ we have

$$\int \varphi(x) f * f^\sim(x)\, \mu(\mathrm{d}x) = \int \left(f * f^\sim(x) \int \chi(x)\, \beta_\varphi(\mathrm{d}\chi) \right) \mu(\mathrm{d}x)$$
$$= \int \left(\int \chi f * (\chi f)^\sim \, \mathrm{d}\mu \right) \beta_\varphi(\mathrm{d}\chi)$$
$$= \int \left(\int |\overline{\mathcal{F}}_G(\chi f)|^2 \, \mathrm{d}\beta_\mu \right) \beta_\varphi(\mathrm{d}\chi)$$
$$= \iint |\overline{\mathcal{F}}_G f|^2(\chi + \varrho)\, \beta_\mu(\mathrm{d}\varrho)\, \beta_\varphi(\mathrm{d}\chi),$$

and since the last double integral is $\geqslant 0$, $\varphi \cdot \mu \in M_p(G)$. The rest follows from Theorem 4.4.11.

4.4.13.3 Given $\mu \in M_p(G)$ with generalized Bochner measure β_μ we have that $T_\chi(\beta_\mu) = \beta_\mu$ for all $\chi \in G^\wedge$ if and only if $\chi = 1$ on $\mathrm{supp}(\mu)$ for all $\chi \in G^\wedge$.

In fact, for $\chi \in G^\wedge$, $\beta_\chi = \varepsilon_\chi$, hence $\beta_{\chi \cdot \mu} = \varepsilon_\chi * \beta_\mu = T_\chi \beta_\mu$ by Consequence 4.4.13.2. But now Corollary 4.4.12 applies and yields that $T_\chi(\beta_\mu) = \beta_\mu$ if and only if $\chi \cdot \mu = \mu$ holds. This statement, however, is equivalent to $\chi = 1$ on $\mathrm{supp}(\mu)$.

4.4.13.4 *Since* $\varepsilon_0 \in M_{\mathrm{p}}(G)$, *Consequence 4.4.13.3 implies that*

$$\beta_{\varepsilon_0} = \omega_{G^\wedge} ,$$

hence by Theorem 4.4.10 that

$$f * f^\sim(x) = \int \chi(x) |\hat{f}(\chi)|^2 \, \beta_\mu(\mathrm{d}\chi)$$

for all $f \in C^c(G)$ *and* $x \in G$. *In particular, for* $x = 0$ *we obtain*

$$\int |f(x)|^2 \, \omega_G(\mathrm{d}x) = \int |\hat{f}(\chi)|^2 \, \beta_\mu(\mathrm{d}\chi)$$

and hence that $\beta_\mu = \omega_{G^\wedge}$.

We have regained the classical version of the Plancherel theorem.

Theorem 4.4.14 *For any measure* $\mu \in M^{\mathrm{b}}(G)$ *the following statements are equivalent:*

(i) $\mu \in M_{\mathrm{p}}(G)$.
(ii) $\hat{\mu}(\chi) \geqslant 0$ *for all* $\chi \in G^\wedge$.

If any of these equivalent conditions is satisfied then

$$\beta_\mu = \hat{\mu} \cdot \omega_{G^\wedge} .$$

Proof.   (i) $\Rightarrow$ (ii). For $\mu \in M^{\mathrm{b}}(G)$ we have $|\hat{\mu}(\chi)| \leqslant \|\mu\|$, hence

$$\int |\hat{f}(\chi)|^2 |\hat{\mu}(\chi)| \, \omega_{G^\wedge}(\mathrm{d}\chi) \leqslant \|\mu\| \int |\hat{f}(\chi)|^2 \, \omega_{G^\wedge}(\mathrm{d}\chi) < \infty$$

whenever $f \in C^c(G)$. From Consequence 4.4.13.4 (with β_μ replaced by $\omega_{G^\wedge}$) we infer that for all $f \in C^c(G)$ and $x \in G$

$$\mu * f * f^\sim(x) = \int \chi(x) |\hat{f}(\chi)|^2 \hat{\mu}(\chi) \, \omega_{G^\wedge}(\mathrm{d}\chi) \tag{4}$$

holds. Thus the measure $\hat{\mu} \cdot \omega_{G^\wedge}$ fulfills the conditions (i) and (ii) of Theorem 4.4.10 since $\mu \in M_{\mathrm{p}}(G)$, $\beta_\mu = \hat{\mu} \cdot \omega_{G^\wedge}$ and therefore $\beta_\mu \geqslant 0$. It follows that $\hat{\mu} \geqslant 0$.

(ii) $\Rightarrow$ (i). If conversely $\hat{\mu} \geqslant 0$ then (4) implies that

$$\int f * f^\sim \, \mathrm{d}\mu = \int |\overline{\mathcal{F}}_G f(\chi)|^2 \hat{\mu}(\chi) \, \omega_{G^\wedge}(\mathrm{d}\chi) \geqslant 0$$

for all $f \in C^c(G)$, hence that $\mu \in M_{\mathrm{p}}(G)$. $\square$

Definition 4.4.15 For any measure $\mu \in M_{\mathrm{p}}(G)$ with associated Bochner measure $\beta_\mu \in M_+(G^\wedge)$ the **generalized Fourier transform** of μ is given by

$$\mathcal{F}(\mu) = \mathcal{F}_G(\mu) := \beta_\mu \, .$$

Clearly, the **generalized Fourier mapping** $\mathcal{F} := \mathcal{F}_G : M_{\mathrm{p}}(G) \to M_+(G^\wedge)$ is additive, positive homogeneous and injective, the latter property following from Corollary 4.4.12.

In analogy to Theorem 4.3.7 we now prove

Theorem 4.4.16 $\mathcal{F}_G$ *is a homeomorphism of the cone* $(M_{\mathrm{p},+}(G), \tau_{\mathrm{v}})$ *onto the cone* $(M_{\mathrm{p},+}(G^\wedge), \tau_{\mathrm{v}})$ *with* $\mathcal{F}_G^{-1} = \mathcal{F}_{G^\wedge}$.

Proof. 1. $\mathcal{F}_G$ maps $\mathrm{CPD}_+(G)$ into $M_{\mathrm{p},+}(G^\wedge)$, and

$$\mathrm{Res}_{\mathrm{CPD}_+(G)} \mathcal{F}_{G^\wedge} \mathcal{F}_G = \mathrm{id}_{\mathrm{CPD}_+(G)} \, .$$

In fact, let $\varphi \in \mathrm{CPD}_+(G)$ with Bochner measure $\beta := \mathcal{F}_G \varphi$. Then

$$\mathcal{F}_{G^\wedge} \beta(x) = \overline{\mathcal{F}}_{G^\wedge} \beta(-x) = \varphi(-x) = \varphi(x)$$

whenever $x \in G$. Theorem 4.4.14 implies that $\beta \in M_{\mathrm{p},+}(G^\wedge)$, hence the assertion follows.

2. Now we show that

$$\mathcal{F}_G M_{\mathrm{p},+}(G) \subset M_{\mathrm{p},+}(G^\wedge) \, .$$

Let $\mu \in M_{\mathrm{p},+}(G)$. Given an approximate identity $\{\psi_U : U \in \mathfrak{U}\}$ in $C^{\mathrm{c}}(G)$ for which necessarily

$$\psi_U \cdot \omega_G \xrightarrow{\tau_{\mathrm{w}}} \varepsilon_0$$

or

$$\hat{\psi}_U \xrightarrow{\tau_{\mathrm{co}}} 1 \quad \text{as} \quad U \to \{0\}$$

holds (Theorem 4.3.7), for every $U \in \mathfrak{U}$ the function $\psi_{U,\mu} := \mu * \psi_U * \psi_{\widetilde{U}}$ belongs to $\mathrm{CPD}_+(G)$, hence

$$\mathcal{F}_G \psi_{U,\mu} = |\widehat{\psi}_U|^2 \cdot \mathcal{F}_G \mu \in M_{\mathrm{p},+}(G^\wedge) \, .$$

This implies that

$$\int g * g^\sim |\widehat{\psi}_U|^2 \, \mathrm{d}(\mathcal{F}_G \mu) \geqslant 0$$

for all $g \in C^c(G^\wedge)$ and all $U \in \mathfrak{U}$. For $U \to \{0\}$ this yields

$$\int g * g^\sim \, \mathrm{d}(\mathcal{F}_G \mu) \geqslant 0$$

for all $g \in C^c(G^\wedge)$, and this shows that $\mathcal{F}_G \mu \in M_{\mathrm{p},+}(G^\wedge)$.

3. Next we prove that $\mathcal{F}_{G^\wedge} \mathcal{F}_G = \mathrm{id}$ (on $M_{\mathrm{p},+}(G)$). Let $\mu \in M_{\mathrm{p},+}(G)$, $\{\psi_U : U \in \mathfrak{U}\}$ and $\psi_{U,\mu}$ $(U \in \mathfrak{U})$ be as in part 2. of this proof. Applying Consequence 4.4.13.2 with G replaced by $G^\wedge$ we obtain on the one hand

$$\mathcal{F}_{G^\wedge}(\mathcal{F}_G \psi_{U,\mu}) = \mathcal{F}_{G^\wedge}\left(|\hat{\psi}_U|^2 \cdot \mathcal{F}_G \mu\right)$$
$$= \psi_U * \psi_{\widetilde{U}} * \mathcal{F}_{G^\wedge} \mathcal{F}_G \mu,$$

on the other hand with the help of part 1. of this proof

$$\mathcal{F}_{G^\wedge}(\mathcal{F}_G \psi_{U,\mu}) = \psi_{U,\mu}$$
$$= \psi_U * \psi_{\widetilde{U}} * \mu,$$

since $\psi_{U,\mu} \in \mathrm{CPD}_+(G)$ for all $U \in \mathfrak{U}$. Consequently

$$\psi_U * \psi_{\widetilde{U}} * \mathcal{F}_{G^\wedge} \mathcal{F}_G \mu = \psi_U * \psi_{\widetilde{U}} * \mu$$

for all $U \in \mathfrak{U}$ and in the limit as $U \to \{0\}$ the desired identity $\mathcal{F}_{G^\wedge} \mathcal{F}_G \mu = \mu$.

4. It remains to be shown that $\mathcal{F}_G$ is a homeomorphism. For the purpose of that proof it is sufficient to verify the continuity of the Fourier mapping $\mathcal{F}_G : M_{\mathrm{p},+}(G) \to M_{\mathrm{p},+}(G^\wedge)$. Employing part 3. of this proof and interchanging the roles of G and $G^\wedge$ yields the final statement.

Let $(\mu_\alpha)_{\alpha \in A}$ be a net in $M_{\mathrm{p},+}(G)$ such that

$$\mu_\alpha \xrightarrow{\tau_v} \mu \in M_{\mathrm{p},+}(G).$$

It is easy to see that this implies

$$\mu_\alpha * f * f^\sim \xrightarrow{\tau_{\mathrm{co}}} \mu * f * f^\sim$$

for all $f \in C^c(G)$. By Theorem 4.3.7

$$\tau_{\mathrm{w}} - \lim_{\alpha \in A} |\hat{f}|^2 \mathcal{F}_G \mu_\alpha = \tau_{\mathrm{w}} - \lim_{\alpha \in A} \mathcal{F}_G(\mu_\alpha * f * f^\sim)$$
$$= \mathcal{F}_G(\mu * f * f^\sim)$$
$$= |\hat{f}|^2 \mathcal{F}_G \mu.$$

Now, for any $\psi \in C^c(G^\wedge)$ we choose $f \in C^c(G)$ such that $\hat{f}(\chi) \neq 0$ for all $\chi \in \mathrm{supp}(\psi)$. The function

$$\chi \mapsto h(\chi) := \begin{cases} \dfrac{\psi(\chi)}{|\hat{f}(\chi)|^2} & \text{if } \hat{f}(\chi) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

belongs to $C^c(G^\wedge)$. Then

$$\lim_{\alpha \in A} \int h|\hat{f}|^2 \, \mathrm{d}(\mathcal{F}_G\mu_\alpha) = \int h|\hat{f}|^2 \, \mathrm{d}(\mathcal{F}_G\mu)$$

which implies

$$\lim_{\alpha \in A} \int \psi \, \mathrm{d}(\mathcal{F}_G\mu_\alpha) = \int \psi \, \mathrm{d}(\mathcal{F}_G\mu)$$

and hence that

$$\mathcal{F}_G\mu_\alpha \xrightarrow{\tau_v} \mathcal{F}_G\mu \, .$$

$\square$

On the way of showing that Haar measures ω_H of closed subgroups H of G are positive definite we are starting by studying the invariance set of a measure on G.

Definition 4.4.17 A measure $\mu \in M(G)$ is said to be a-**invariant** with invariance point $a \in G$ if

$$\mu * \varepsilon_a = \mu \, .$$

The totality of all invariance points of μ which obviously is a closed subgroup of G, will be called the **invariance group** of μ.

Both notions are also employed for functions $f \in C(G)$ by considering f as the measure $f\omega_G$.

The invariance groups of μ and f are denoted by $\mathrm{Inv}(\mu)$ and $\mathrm{Inv}(f)$ respectively.

Finally, $\mu \in M(G)$ is said to be H-**invariant** for some subset H of G if $H \subset \mathrm{Inv}(\mu)$.

Clearly, for any closed subgroup H of G Haar measure ω_H of H (viewed as a measure in $M_+(G)$) is H-invariant (with $\mathrm{supp}(\omega_H) = H$). If $\mu \in M^b(G) \setminus \{0\}$ then $\mathrm{Inv}(\mu)$ is a compact subgroup of G.

In fact, there exists a function $f \in C^c(G)$ such that $g := \mu * f \neq 0$. For any $x_0 \in G$ with $g(x_0) \neq 0$ we obtain the inclusion

$$x_0 + \mathrm{Inv}(\mu) \subset \{x \in G : g(x) = g(x_0)\} \, .$$

Since $M^b(G) \subset M^\infty(G)$, the set $\{x \in G : g(x) = g(x_0)\}$ is compact, hence $\mathrm{Inv}(\mu)$ is compact.

Properties 4.4.18 of invariance groups.

4.4.18.1 *For measures* $\mu \in M^b(G)$ *we have that*

$$\mathrm{Inv}(\mu) = (\mathrm{supp}(\hat{\mu}))^\perp$$

and

$$\mathrm{Inv}(\hat\mu) = (\mathrm{supp}(\mu))^\perp \,.$$

We only argue in favour of the first equality. By the injectivity of the Fourier mapping (Property 4.2.16.3) $a \in G$ is an invariance point of μ if and only if

$$\overline{\chi(a)}\hat\mu(\chi) = \hat\mu(\chi)$$

for all $\chi \in G^\wedge$ and this in turn holds if and only if $\chi(a) = 1$ for all $\chi \in \mathrm{supp}(\hat\mu)$ which says that $a \in (\mathrm{supp}(\hat\mu))^\perp$.

4.4.18.2 *For functions* $\varphi \in \mathrm{CPD}(G)$ *with Bochner measure* $\beta_\varphi \in M^{\mathrm{b}}_+(G^\wedge)$ *we have that*

$$\mathrm{Inv}(\varphi) = \{x \in G : \varphi(x) = \varphi(0)\}$$
$$= (\mathrm{supp}(\beta_\varphi))^\perp \,.$$

At first we infer from Property 4.4.18.1 that

$$\mathrm{Inv}(\varphi) = (\mathrm{supp}(\beta^*_\varphi))^\perp$$
$$= (\mathrm{supp}(\beta_\varphi))^\perp \,.$$

Now, any invariance point of φ satisfies $\varphi(x) = \varphi(0)$. On the other hand, if $\varphi(x) = \varphi(0)$ for $x \in G$ then Property 4.3.2.3 implies that

$$\varphi(x + y) = \varphi(y)$$

whenever $y \in G$. But this yields $x \in \mathrm{Inv}(\varphi)$.

4.4.18.3 *For measure* $\mu \in M_{\mathrm{p}}(G)$ *we have that*

$$\mathrm{Inv}(\mu) = (\mathrm{supp}(\mathcal{F}\mu))^\perp$$

and

$$\mathrm{Inv}(\mathcal{F}\mu) = (\mathrm{supp}(\mu))^\perp \,.$$

While the last equality follows from Consequence 4.4.18.3, the first one requires a detailed proof. From formula (ii) of Theorem 4.4.10 modified by polarization we deduce that

$$\mu * \varepsilon_a * f * g^\sim(x) = \int \chi(x)\overline{\chi(a)}\hat{f}(\chi)\overline{\hat{g}(\chi)}\,\mathcal{F}\mu(\mathrm{d}\chi) \tag{5}$$

valid for all $f, g \in C^{\mathrm{c}}(G)$ and $a \in G$. If $a \in \mathrm{Inv}(\mu)$, then this equality implies that the bounded measures $\hat{f}\overline{\hat{g}}\cdot\mathcal{F}\mu$ and $\hat\varepsilon_a\hat{f}\overline{\hat{g}}\cdot\mathcal{F}\mu$ have the same inverse Fourier transforms and hence are equal. But then $\chi(a) = 1$ for all $\chi \in \mathrm{supp}(\mathcal{F}\mu)$ and hence $a \in (\mathrm{supp}(\mathcal{F}\mu))^\perp$. If, conversely, $a \in G$ satisfies $\chi(a) = 1$ for all $\chi \in \mathrm{supp}(\mathcal{F}\mu)$, then (5) implies that

$$\mu * f * g^\sim(x) = \mu * \varepsilon_a * f * g^\sim(x)$$

holds for all $f, g \in C^{\mathrm{c}}(G)$ and all $x \in G$. Letting f and g run through an approximate identity in $C^{\mathrm{c}}(G)$ we achieve $\mu = \mu * \varepsilon_a$ or $a \in \mathrm{Inv}(\mu)$ as desired.

Now, let H be a closed subgroup of the given locally compact Abelian group G, and let π denote the canonical homomorphism from G onto the quotient group G/H. For every H-invariant function $f \in C(G)$ there exists exactly one quotient function $\dot{f} \in C(G/H)$ such that $\dot{f} \circ \pi = f$. Let ω_H be a fixed Haar measure of H. Since, for any $f \in C^c(G)$ the function $\omega_H * f$ is H-invariant, its quotient function $(\omega_H * f)^{\cdot}$ is uniquely determined and an element of $C^c(G/H)$. Consequently we obtain a mapping $\sigma : C^c(G) \to C^c(G/H)$ defined by

$$\sigma(f) := (\omega_H * f)^{\cdot}$$

for all $f \in C^c(G)$ which is linear, positive and continuous (with respect to the topology τ_{co}). The transpose $\sigma^\top$ of σ maps $M(G/H)$ linearly into the set $M(G,H)$ of H-invariant measures in $M(G)$.

Properties 4.4.19 of the mapping σ.

4.4.19.1 σ is a surjection from $C^c(G)$ onto $C^c(G/H)$.

4.4.19.2 σ satisfies the invariance condition

$$\sigma(T_a f) = T_{\pi(a)} \sigma(f)$$

valid for all $f \in C^c(G)$, $a \in G$.

4.4.19.3 Given Haar measures ω_G and ω_H of G and H respectively there exists a unique Haar measure $\omega_{G/H}$ of G/H such that

$$\sigma^\top(\omega_{G/H}) = \omega_G \,.$$

4.4.19.4 Let $\mu \in M_+(G,H)$. Then there exists a unique **quotient measure** $\dot{\mu} \in M_+(G/H)$ (**associated with** μ) such that

$$\sigma^\top(\dot{\mu}) = \mu \,.$$

More generally,

4.4.19.5 $\sigma^\top$ is an isomorphism from $M(G/H)$ onto $M(G,H)$.

While Properties 4.4.19.2 through 4.4.19.4 follow from obvious computations, Property 4.4.19.1 requires a proof. Let $h \in C^c_+(G/H)$. Since π is a proper mapping, there exists a compact set $K \subset G$ with $\pi(K) = \text{supp}(h)$. Now we choose $\psi \in C^c(G)$ with the property that $\psi = 1$ on K and define

$$f(x) := \begin{cases} \frac{\psi(x)\, h \circ \pi(x)}{\omega_H * \psi(x)} & \text{if } \omega_H * \psi(x) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \in C^c_+(G)$, hence

$$\omega_H * f(x) = \begin{cases} h \circ \pi(x) & \text{if } \omega_H * \psi(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and consequently $\sigma(f) = h$.

4.4.19.6 *Given the Haar measure ω_G of G and a fixed Haar measure ω_H of the closed subgroup H of G there exists a Haar measure $\omega_{G/H}$ of G/H satisfying*

$$\sigma^\top(\omega_{G/H}) = \omega_G .$$

4.4.19.7 *For a closed subgroup H of G any Haar measure ω_H of H belongs to $M_{\mathrm{p}}(G)$, and the generalized Fourier transform $\mathcal{F}\omega_H$ of ω_H is a Haar measure of the closed subgroup $H^\perp$ of $G^\wedge$.*

In fact, for any $f \in C^c(G)$ we obtain from Property 4.4.19.6 that

$$\int f * f^\sim \, \mathrm{d}\omega_H = \int (\omega_H * \overline{f}) f \, \mathrm{d}\omega_G$$

$$= \int |\sigma(f)|^2 \, \mathrm{d}\omega_{G/H} ,$$

hence that $\omega_H \in M_{\mathrm{p}}(G)$. From Property 4.4.18.3 we infer that $\mathcal{F}_G \omega_H$ is $H^\perp$-invariant. The rest is clear.

Properties 4.4.20 (Special case)
*Let K be a compact subgroup of G and let ω_K be the **normed Haar measure** of K in the sense that $\omega_K \in M^1(G)$. Then*

$$\hat{\omega}_K = \mathbb{1}_{K^\perp} ,$$

and

$$\mathbb{1}_{K^\wedge} \cdot \omega_{G^\wedge} = \mathcal{F}_G \omega_K$$

is a Haar measure of $K^\perp$.

The normed Haar measure ω_K of a compact subgroup K of G will play an important role in the probabilistic implications of this section to be discussed in Chapter 6.

Chapter 5

Negative Definite Functions and Convolution Semigroups

5.1 Negative definite functions

In the present section we are going to study a notion dual to positive definiteness in order to obtain an analytic tool for the description of convolution semigroups of measures on the locally compact Abelian group G with dual group $G^\wedge$.

Definition 5.1.1 A complex-valued function ψ on $G^\wedge$ is called **negative definite** if for all $n \geqslant 1$ and for all $\chi_1, \ldots, \chi_n \in G^\wedge$ the matrix
$$\left(\psi(\chi_i) + \overline{\psi(\chi_j)} - \psi(\chi_i - \chi_j) \right) \in \mathbb{C}^{n \times n}$$
is positive hermitian.

Let $\mathrm{ND}(G^\wedge)$ denote the totality of all negative definite functions on G, and let $\mathrm{CND}(G^\wedge) := \mathrm{ND}(G^\wedge) \cap C(G^\wedge)$.

Properties 5.1.2 of a function $\psi \in \mathrm{ND}(G^\wedge)$.

5.1.2.1 $\psi(0) \geqslant 0$.

5.1.2.2 $\psi^\sim = \psi$ *and* $\operatorname{Re} \psi \geqslant \psi(0)$.

5.1.2.3 $\sqrt{|\psi|}$ *is subadditive.*

For proofs we note that Property 5.1.2.2 follows from the fact that for every $\chi \in G^\wedge$ the matrix
$$\begin{bmatrix} \psi(\chi) + \overline{\psi(\chi)} - \psi(0) & \psi(\chi) + \overline{\psi(0)} - \psi(\chi) \\ \psi(0) + \overline{\psi(\chi)} - \psi(-\chi) & \psi(0) + \overline{\psi(0)} - \psi(0) \end{bmatrix} \in \mathbb{C}^{2 \times 2}$$
is positive hermitian, and that Property 5.1.2.3 is a consequence of the positive hermiteness of the matrix
$$\begin{bmatrix} \psi(\chi) + \overline{\psi(\chi)} - \psi(0) & \psi(\chi) + \overline{\psi(\varrho)} - \psi(\chi - \varrho) \\ \psi(\varrho) + \overline{\psi(\chi)} - \psi(\varrho - \chi) & \psi(\varrho) + \overline{\psi(\varrho)} - \psi(0) \end{bmatrix} \in \mathbb{C}^{2 \times 2}$$

valid for all $\chi, \varrho \in G^\wedge$. In fact, applying that $\psi^\sim = \psi$ one obtains the inequalities

$$|\psi(\chi) + \overline{\psi(\varrho)} - \psi(\chi - \varrho)|^2 \leqslant (2\,\mathrm{Re}\,\psi(\chi) - \psi(0))(2\,\mathrm{Re}\,\psi(\varrho) - \psi(0))$$
$$\leqslant 4|\psi(\chi)\psi(\varrho)|\,,$$

hence

$$|\psi(\chi + \varrho)| \leqslant (\sqrt{|\psi(\chi)|} + \sqrt{|\psi(\varrho)|})^2\,,$$

and this yields the subadditivity of $\sqrt{|\psi|}$.

Properties 5.1.3 of the set $\mathrm{ND}(G^\wedge)$.

5.1.3.1 $\mathrm{ND}(G^\wedge)$ *is closed under formation of complex conjugates and real parts.*

5.1.3.2 *The constant function $\geqslant 0$ belongs to $\mathrm{ND}(G^\wedge)$.*

5.1.3.3 $\mathrm{ND}(G^\wedge)$ *is a τ_p-closed convex cone, $\mathrm{CND}(G^\wedge)$ a τ_co-closed convex cone (in $C(G^\wedge)$).*

The proofs of these properties are obvious.

Theorem 5.1.4 *For any complex-valued function ψ on $G^\wedge$ the following statement are equivalent:*

(i) $\psi \in \mathrm{ND}(G^\wedge)$.

(ii) (a) $\psi(0) \geqslant 0$,

　　(b) $\psi^\sim = \psi$, *and*

　　(c) *for all $n \geqslant 1$, $\chi_1, \ldots, \chi_n \in G^\wedge$ and $c_1, \ldots, c_n \in \mathbb{C}$ with $\sum_{i=1}^n c_i = 0$ one has*

$$\sum_{i,j=1}^n \psi(\chi_i - \chi_j)c_i\overline{c_j} \leqslant 0\,.$$

Proof. (i) $\Rightarrow$ (ii). Given $\psi \in \mathrm{ND}(G^\wedge)$ it remains to show (c). Let $n \geqslant 1$, $\chi_1, \ldots, \chi_n \in G^\wedge$ and $c_1, \ldots, c_n \in \mathbb{C}$ with $\sum_{i=1}^n c_i = 0$. Then

$$0 \leqslant \sum_{i,j=1}^n (\psi(\chi_i) + \overline{\psi(\chi_j)} - \psi(\chi_i - \chi_j))c_i\overline{c_j}$$

$$= \sum_{j=1}^n \overline{c_j}\left(\sum_{i=1}^n \psi(\chi_i)c_i\right) + \sum_{i=1}^n c_i\left(\sum_{j=1}^n \overline{\psi(\chi_j)c_j}\right) - \sum_{i,j=1}^n \psi(\chi_i - \chi_j)c_i\overline{c_j}$$

$$= -\sum_{i,j=1}^n \psi(\chi_i - \chi_j)c_i\overline{c_j}\,.$$

(ii) $\Rightarrow$ (i). Let ψ satisfy (a) to (c) of (ii), and assume given $\chi_1, \ldots, \chi_n \in G^\wedge$, $c_1 \ldots, c_n \in \mathbb{C}$. Considering the sequences $\{0, \chi_1, \ldots, \chi_n\}$ and $\{c, c_1, \ldots, c_n\}$ with $c := -\sum_{i=1}^n c_i$ we obtain from (c) that

$$\psi(0)|c|^2 + \sum_{i=1}^n \psi(\chi_i)c_i\overline{c} + \sum_{j=1}^n \psi(-\chi_j)c\overline{c}_j + \sum_{i,j=1}^n \psi(\chi_i - \chi_j)c_i\overline{c}_j \leqslant 0$$

and with the help of (a) that

$$\sum_{i,j=1}^n (\psi(\chi_i) + \overline{\psi(\chi_j)} - \psi(\chi_i - \chi_j))c_i\overline{c}_j \geqslant \psi(0)|c|^2 \geqslant 0,$$

i.e. (i). $\square$

Further

Properties 5.1.5

5.1.5.1 *If* $\psi \in \mathrm{ND}(G^\wedge)$, *then also* $\psi - \psi(0) \in ND(G^\wedge)$.

In fact, for $\chi_1, \ldots, \chi_n \in G^\wedge$, $c_1, \ldots, c_n \in \mathbb{C}$ with $\sum_{i=1}^n c_i = 0$ we obtain that

$$\sum_{i,j=1}^n (\psi(\chi_i - \chi_j) - \psi(0))c_i\overline{c}_j = \sum_{i,j=1}^n \psi(\chi_i - \chi_j)c_i\overline{c}_j \leqslant 0$$

by assumption. But the function $\psi - \psi(0)$ clearly satisfies the conditions (a) and (b) of (ii) of Theorem 5.1.4. The theorem implies that it belongs to $\mathrm{ND}(G^\wedge)$.

5.1.5.2 *If* $\varphi \in \mathrm{PD}(G^\wedge)$, *then* $\varphi(0) - \varphi \in ND(G^\wedge)$.

Again, we see that for $\chi_1, \ldots, \chi_n \in G^\wedge$ and $c_1, \ldots, c_n \in \mathbb{C}$ with $\sum_{i=1}^n c_i = 0$ the positive definiteness of φ implies that

$$\sum_{i,j=1}^n (\varphi(0) - \varphi(\chi_i - \chi_j))c_i\overline{c}_j = -\sum_{i,j=1}^n \varphi(\chi_i - \chi_j)c_i\overline{c}_j \leqslant 0.$$

Since the function $\varphi(0) - \varphi$ satisfies the conditions (a) and (b) of (ii) of Theorem 5.1.4, $\varphi(0) - \varphi$ turns out to belong to $\mathrm{ND}(G^\wedge)$.

The following result describes the connection between the sets $\mathrm{PD}(G^\wedge)$ and $\mathrm{ND}(G^\wedge)$.

Theorem 5.1.6 (Schoenberg duality, noncontinuous case)
For any complex-valued function ψ *on* $G^\wedge$ *the subsequent statements are equivalent:*

(i) $\psi \in \mathrm{ND}(G^\wedge)$.

(ii) (a) $\psi(0) \geqslant 0$ *and*
    (b) $\exp(-t\psi) \in \mathrm{PD}(G^\wedge)$ *for all* $t > 0$.

Proof. (i) $\Rightarrow$ (ii). Given $\psi \in \mathrm{ND}(G^\wedge)$ it suffices to show that $\exp(-\psi) \in \mathrm{PD}(G^\wedge)$. Let therefore $\chi_1, \ldots, \chi_n \in G^\wedge$. Since the matrix

$$\left(\psi(\chi_i) + \overline{\psi(\chi_j)} - \psi(\chi_i - \chi_j)\right) \in \mathbb{C}^{n \times n}$$

is positive hermitian, also the matrix

$$\left(\exp(\psi(\chi_i) + \overline{\psi(\chi_j)} - \psi(\chi_i - \chi_j))\right)$$

is positive hermitian. But then we obtain for $c_1, \ldots, c_n \in \mathbb{C}$ that

$$\sum_{i,j=1}^{n} \exp(-\psi(\chi_i - \chi_j)) c_i \overline{c}_j$$

$$= \sum_{i,j=1}^{n} \exp(\psi(\chi_i) + \overline{\psi(\chi_j)} - \psi(\chi_i - \chi_j)) \exp(-\psi(\chi_i)) \exp(-\overline{\psi(\chi_j)}) c_i \overline{c}_j$$

$$= \sum_{i,j=1}^{n} \exp(\psi(\chi_i) + \overline{\psi(\chi_j)} - \psi(\chi_i - \chi_j)) d_i \overline{d}_j \geqslant 0 \,,$$

where $d_i := \exp(-\psi(\chi_i)) c_i \in \mathbb{C}$ for $i = 1, \ldots, n$, and this is the assertion.

(ii) $\Rightarrow$ (i). Suppose now that ψ satisfies (a) and (b) of (ii). From (a) we infer that $\exp(-t\psi(0)) \leqslant 1$ for all $t > 0$, hence by Property 5.1.5.2 that

$$\frac{1}{t}(1 - \exp(-t\psi)) \in ND(G^\wedge)$$

for all $t > 0$. Moreover we have that

$$\psi = \lim_{t \to 0} \frac{1}{t}(1 - \exp(-t\psi))$$

on $G^\wedge$, hence by Property 5.1.3.3 that $\psi \in \mathrm{ND}(G^\wedge)$. $\square$

The Schoenberg duality permits to prove two more important

Properties 5.1.7 of functions $\psi \in \mathrm{ND}(G^\wedge)$.

5.1.7.1 (Forming the inverse) *If* $\psi(0) > 0$ *then* $\frac{1}{\psi} \in \mathrm{PD}(G^\wedge)$.

By Theorem 5.1.6 the function $\exp(-t\psi)$ belongs to $\mathrm{PD}(G^\wedge)$ for all $t > 0$. Moreover,

$$|\exp(-t\psi)| \leqslant \exp(-t\psi(0))$$

for all $t > 0$. Consequently

$$\frac{1}{\psi} = \int_0^\infty \exp(-t\psi)\,\mathrm{d}t \in \mathrm{PD}(G^\wedge)\,.$$

5.1.7.2 (Approximation) *There exist sequences* $(a_n)_{n\geqslant 1}$ *in* $\mathbb{R}_+$ *and* $(\varphi_n)_{n\geqslant 1}$ *in* $\mathrm{PD}(G^\wedge)$ *such that for the sequence* $(\psi_n)_{n\geqslant 1}$ *with*

$$\psi_n := a_n + \varphi_n(0) - \varphi_n \qquad (n \geqslant 1)$$

the limit relationship

$$\psi = \tau_\mathrm{p} - \lim_{n\to\infty} \psi_n$$

holds (on $G^\wedge$*).*

In fact, considering the sequence $(\varphi_n)_{n\geqslant 1}$ of functions

$$\varphi_n := n\exp(-\frac{1}{n}(\psi - \psi(0)))$$

which by Theorem 5.1.6 belong to $\mathrm{PD}(G^\wedge)$, and the sequence $(a_n)_{n\geqslant 1}$ of numbers $a_n := \psi(0) \in \mathbb{R}_+$ we obtain that for any $\chi \in G^\wedge$

$$\psi(\chi) - \psi_n(\chi) = \frac{1}{n}\left\{\frac{(\psi(\chi) - \psi(0))^2}{2!} - \frac{(\psi(\chi) - \psi(0))^3}{3!} + \cdots\right\}$$

hence

$$|\psi(\chi) - \psi_n(\chi)| \leqslant \frac{1}{n}\exp(|\psi(\chi) - \psi(0)|)$$

holds.

We note that given sequences $(a_n)_{n\geqslant 1}$ in $\mathbb{R}_+$ and $(\varphi_n)_{n\geqslant 1}$ in $\mathrm{PD}(G^\wedge)$ the sequence $(\psi_n)_{n\geqslant 1}$ defined in Property 5.1.7.2 clearly belongs to $\mathrm{ND}(G^\wedge)$ as follows from Properties 5.1.3 and 5.1.5.2.

Examples 5.1.8 of functions in $\mathrm{ND}(G^\wedge)$.

5.1.8.1 Any homomorphism h from $G^\wedge$ into $\mathbb{R}$ belongs to $\mathrm{ND}(G^\wedge)$. Moreover,

5.1.8.2 given a real function h on $G^\wedge$ the function $\psi := ih$ belongs to $\mathrm{ND}(G^\wedge)$ if and only if h is a homomorphism.

In fact, for all $t > 0$ the function

$$\chi \mapsto \varphi_t(\chi) := \exp(-t\psi(\chi))$$

is positive definite with $|\varphi_t| = 1$. But then from Example 4.3.4.2 we conclude that $\varphi_t \in G^\wedge$ and therefore

$$\exp(-ith(\chi + \varrho)) = \exp(-it(h(\chi) + h(\varrho)))$$

which implies that

$$h(\chi + \varrho) = h(\chi) + h(\varrho)$$

whenever $\chi, \varrho \in G^{\wedge}$.

5.1.8.3 Any quadratic form $q \geqslant 0$ on $G^{\wedge}$ belongs to $\mathrm{ND}(G^{\wedge})$.

We recall that quadratic forms q on $G^{\wedge}$ defined by

$$q(\chi + \varrho) + q(\chi - \varrho) = 2[q(\chi) + \varrho(\varrho)]$$

for all $\chi, \varrho \in G^{\wedge}$ have the properties $q(0) = 0$, $q(\chi) = q(-\chi)$ and $q(n\chi) = n^2 q(\chi)$ whenever $\chi \in G^{\wedge}$ and $n \in \mathbb{N}$. The mapping $Q : G^{\wedge} \times G^{\wedge} \to \mathbb{R}$ given by

$$Q(\chi, \varrho) := q(\chi) + q(\varrho) - q(\chi - \varrho)$$

for all $\chi, \varrho \in G^{\wedge}$ is non-negative, symmetric and additive in both variables the latter property following from the subsequent sequence of equalities (related to the first variable):

$$q(\chi) + q(\varrho) - q(\chi - \varrho) + q(\sigma) + q(\varrho) - q(\sigma - \varrho)$$

$$= q(\chi) + q(\sigma) + 2q(\varrho) - \frac{1}{2}[q(\chi - \varrho + \sigma - \varrho) + q(\chi - \varrho - (\sigma - \varrho))]$$

$$= q(\chi) + q(\sigma) - \frac{1}{2}q(\chi - \sigma) + 2q(\varrho)$$

$$\quad - \frac{1}{2}[2q(\chi + \sigma - \varrho) + 2q(\varrho) - q(\chi + \sigma)]$$

$$= \frac{1}{2}q(\chi + \sigma) + 2q(\varrho) - q(\chi + \sigma - \varrho) - q(\varrho) + \frac{1}{2}q(\chi + \sigma)$$

$$= q(\chi + \sigma) + q(\varrho) - q(\chi + \sigma - \varrho),$$

χ, ϱ, σ being taken from $G^{\wedge}$. But now

$$\sum_{i,j=1}^{n} \left[q(\chi_i) + \overline{q(\chi_j)} - q(\chi_i - \chi_j)\right] c_i \overline{c}_j = Q(\chi, \chi) \geqslant 0$$

for $n \geqslant 1$, $\chi_1, \ldots, \chi_n \in G^{\wedge}$, $c_1, \ldots, c_n \in \mathbb{Z}$ and $\chi := \sum_{i=1}^{n} c_i \chi_i$, and this suffices to see that $q \in \mathrm{ND}(G^{\wedge})$.

For later application we note that

5.1.8.4 given a constant $c \geqslant 0$, a homomorphism $h : G^{\wedge} \to \mathbb{R}$ and a quadratic form $q \geqslant 0$ on $G^{\wedge}$ the function

$$\psi := c + \mathrm{i}h + q$$

belongs to $\mathrm{ND}(G^{\wedge})$.

This follows from Properties 5.1.3.

5.2   Convolution semigroups and resolvents

We are now prepared to discuss the representation of continuous negative definite functions on $G^\wedge$ in terms of measures on G. For this purpose we introduce the subset

$$M_+^{(a)}(G) := \{\mu \in M_+^{\mathrm{b}}(G) : \|\mu\| \leqslant a\}$$

of nonnegative measures on G bounded by $a > 0$.

Definition 5.2.1 A (one-parameter) family $(\mu_t)_{t>0}$ of measures in $M_+^{(1)}(G)$ is called a **semigroup** (of measures) **in** $M_+^{(1)}(G)$ if the mapping $t \mapsto \mu_t$ from $\mathbb{R}_+^\times$ into $M_+^{(1)}(G)$ is a homomorphism (of semigroups).

$(\mu_t)_{t>0}$ is said to be τ_{v}- or τ_{w}-**continuous** if this homomorphism is τ_{v}- or τ_{w}-continuous respectively. In the case of τ_{v}-continuity

$$\mu_0 := \tau_{\mathrm{v}} - \lim_{t \to 0} \mu_t$$

exists, and μ_0 is an idempotent measure in $M_+^{(1)}(G)$.

$(\mu_t)_{t>0}$ is called a **convolution semigroup in** $M_+^{(1)}(G)$ if it is τ_{v}-continuous and if

$$\mu_0 = \varepsilon_0 \,.$$

We then write $(\mu_t)_{t\geqslant 0}$ instead of $(\mu_t)_{t>0}$.

Remark 5.2.2 For a convolution semigroup $(\mu_t)_{t\geqslant 0}$ in $M_+^{(1)}(G)$ the τ_{v}- and τ_{w}-continuities coincide.

In fact, let $(\mu_t)_{t\geqslant 0}$ be τ_{v}-continuous. Then for $f \in C^{\mathrm{c}}(G)$ with $0 \leqslant f \leqslant 1$ and $f(0) = 1$ one sees that

$$
\begin{aligned}
1 = f(0) &= \lim_{t \to 0} \int f \, \mathrm{d}\mu_t \\
&\leqslant \liminf_{t \to 0} \|\mu_t\| \\
&\leqslant \limsup_{t \to 0} \|\mu_t\| \leqslant 1 \,,
\end{aligned}
$$

hence by Proposition 4.1.3 that

$$\tau_{\mathrm{w}} - \lim_{t \to 0} \mu_t = \varepsilon_0 \,.$$

For $t, t_0 > 0$ and $\chi \in G^\wedge$ we have that

$$|\hat{\mu}_t(\chi) - \hat{\mu}_{t_0}(\chi)| \leqslant |\hat{\mu}_{|t-t_0|}(\chi) - 1|$$

holds. The above limit relationship together with a double application of Theorem 4.3.7 for $t \to t_0$ yields the assertion.

By $S(G)$ and $CS(G)$ we shall denote the sets of semigroups and convolution semigroups in $M_+^{(1)}(G)$ respectively.

Theorem 5.2.3 (Schoenberg correspondence)
There is a one-to-one correspondence

$$(\mu_t)_{t \geqslant 0} \mapsto \psi$$

between the sets $CS(G)$ *and* $CND(G^\wedge)$ *given by*

$$\hat{\mu}_t = \exp(-t\psi)$$

for all $t \geqslant 0$.

In the situation of this correspondence $(\mu_t)_{t \geqslant 0}$ *and* ψ *are said to be* **associated** *(with each other). Symbolically we shall write*

$$(\mu_t)_{t \geqslant 0} \leftrightarrow \psi \,.$$

Proof. 1. Let $(\mu_t)_{t \geqslant 0} \in CS(G)$ and define for any fixed $\chi \in G^\wedge$ the complex-valued function φ_χ by

$$\varphi_\chi(t) := \hat{\mu}_t(\chi)$$

whenever $t > 0$. By Remark 5.2.2 φ_χ is continuous and satisfies

$$\varphi_\chi(t + u) = \varphi_\chi(t)\, \varphi_\chi(u)$$

for $t, u \in \mathbb{R}_+^\times$ as well as

$$\lim_{t \to 0} \varphi_\chi(t) = 1 \,.$$

Consequently φ_χ is of the form

$$\varphi_\chi = \exp(-t\psi(\chi))$$

for some unique $\psi(\chi) \in \mathbb{C}$ $(t > 0)$. Clearly the function

$$\chi \mapsto \psi(\chi)$$

satisfies $\psi(0) \geqslant 0$, and the function

$$\chi \mapsto \exp(-t\psi(\chi)) = \hat{\mu}_t(\chi)$$

belongs to $CPD(G^\wedge)$ for all $t > 0$. From Schoenberg's duality theorem 5.1.6 we now infer that $\psi \in ND(G^\wedge)$. In order to show that ψ is indeed continuous we look at the equalities

$$\int_0^\infty e^{-t} \hat{\mu}_t(\chi)\, dt = \int_0^\infty \exp(-t(1 + \psi(\chi)))\, dt$$

$$= \frac{1}{1 + \psi(\chi)}$$

valid for all $\chi \in G^\wedge$ and note that the left hand integral of the first equality is the Fourier transform of the Radon measure

$$f \mapsto \int_0^\infty e^{-t} \left(\int f \, d\mu_t \right) dt$$

in $M_+^{(1)}(G)$, and as such it is continuous.

2. Conversely, let $\psi \in \mathrm{CND}(G^\wedge)$. Then for every $t > 0$ the function $\exp(-t\psi)$ belongs to $\mathrm{CPD}(G^\wedge)$ by Theorem 5.1.6. From Bochner's theorem 4.3.5 we conclude that there exists a measure $\mu_t \in M_+^b(G)$ satisfying

$$\hat{\mu}_t = \exp(-t\psi)$$

for each $t > 0$. We shall show that $(\mu_t)_{t \geqslant 0} \in \mathrm{CS}(G)$. First of all $\psi(0) \geqslant 0$ implies that $\|\mu_t\| \leqslant 1$ for all $t \geqslant 0$. Next we conclude from the equalities

$$\begin{aligned}
\hat{\mu}_t(\chi)\,\hat{\mu}_s(\chi) &= \exp(-t\psi(\chi))\,\exp(-s\psi(\chi)) \\
&= \exp(-(t+s)\psi(\chi)) \\
&= \hat{\mu}_{t+s}(\chi)
\end{aligned}$$

valid for all $\chi \in G^\wedge$ that $t \mapsto \mu_t$ is a homomorphism from $\mathbb{R}_+^\times$ into $M_+^{(1)}(G)$, as follows from the injectivity of the Fourier transform stated as Property 4.2.16.3. Since ψ is continuous, hence bounded on compact sets, we obtain that

$$\lim_{t \to 0} \hat{\mu}_t = \lim_{t \to 0} \exp(-t\psi) = 1$$

with respect to the topology τ_{co}. But Theorem 4.3.7 yields

$$\tau_{\mathrm{w}}\!-\!\lim_{t \to 0} \mu_t = \varepsilon_0$$

which implies the assertion. $\qquad\square$

Corollary 5.2.4 (Schoenberg duality, continuous case)
For any complex-valued function ψ on $G^\wedge$ the following statements are equivalent:

(i) $\psi \in \mathrm{CND}(G^\wedge)$.
(ii) (a) $\psi(0) \geqslant 0$ *and*
    (b) $\exp(-t\psi) \in \mathrm{CPD}(G^\wedge)$ *for all* $t > 0$.

Corollary 5.2.5 *Let* $(\mu_t)_{t \geqslant 0} \in \mathrm{CS}(G)$ *and* $\psi \in \mathrm{CND}(G^\wedge)$ *with*

$$(\mu_t)_{t \geqslant 0} \leftrightarrow \psi\,.$$

Then

(i) $\|\mu_t\| = \exp(-t\psi(0))$ *for all* $t > 0$.

In particular,

(ii) $\mu_t \in M^1(G)$ *for all* $t \geqslant 0$ *if and only if* $\psi(0) = 0$.

The **proofs** of these corollaries are clear.

Application 5.2.6 (Generation of convolution semigroups)

5.2.6.1 (Symmetry) *A semigroup* $(\mu_t)_{t>0} \in \mathrm{S}(G)$ *is said to be **symmetric** if* $\mu_t = \mu_t^{\sim}$ *for all* $t > 0$. *Let* $(\mu_t)_{t\geqslant 0} \in \mathrm{CS}(G)$ *and* $\psi \in \mathrm{CND}(G^{\wedge})$ *with*

$$(\mu_t)_{t\geqslant 0} \leftrightarrow \psi \,.$$

Then

$$(\mu_t^{\sim})_{t\geqslant 0} \leftrightarrow \overline{\psi} \,.$$

Clearly $(\mu_t)_{t\geqslant 0} \in \mathrm{CS}(G)$ *is symmetric if and only if* ψ *is real. For any* $(\mu_t)_{t\geqslant 0} \in \mathrm{CS}(G)$ *the convolution semigroup* $(\mu_t * \mu_t^{\sim})_{t\geqslant 0}$ *is symmetric.*

If $(\mu_t)_{t\geqslant 0}$ *is symmetric, then* $\mu_t \in M_{\mathrm{p},+}(G)$, *since*

$$\mu_t = \mu_{\frac{t}{2}} * \mu_{\frac{t}{2}}^{\sim}$$

for all $t \geqslant 0$.

5.2.6.2 (Convolutions) *For* $i = 1, 2$ *let* $(\mu_t^{(i)})_{t\geqslant 0} \in \mathrm{CS}(G)$, $\psi_i \in \mathrm{CND}(G^{\wedge})$ *with*

$$(\mu_t^{(i)})_{t\geqslant 0} \leftrightarrow \psi_i$$

and let $\alpha_i \in \mathbb{R}_+^{\times}$. *Then*

$$\alpha_1\psi_1 + \alpha_2\psi_2 \leftrightarrow (\mu_{\alpha_1 t}^{(1)} * \mu_{\alpha_2 t}^{(2)})_{t\geqslant 0} \,.$$

5.2.6.3 (Products) *For* $i = 1, 2$ *let* G_i *be a locally compact Abelian group with dual* $G_i^{\wedge}$, $(\mu_t^{(i)})_{t\geqslant 0} \in \mathrm{CS}(G_i)$, $\psi_i \in \mathrm{CND}(G_i^{\wedge})$ *such that*

$$\psi_i \leftrightarrow (\mu_t^{(i)})_{t>0} \,.$$

Then the function ψ *on* $G_1^{\wedge} \times G_2^{\wedge}$ *defined by*

$$\psi(\chi_1, \chi_2) := \psi_1(\chi_1) + \psi_2(\chi_2)$$

for all $(\chi_1, \chi_2) \in G_1^{\wedge} \times G_2^{\wedge}$ *belongs to* $\mathrm{CND}(G_1^{\wedge} \times G_2^{\wedge})$, *and*

$$\psi \quad \leftrightarrow \quad \mu_t^{(1)} \otimes \mu_t^{(2)} \,.$$

Examples 5.2.7 of convolution semigroups.

5.2.7.1 A semigroup in $M_+^{(1)}(G)$ is called a **translation semigroup** in $M_+^{(1)}(G)$ if there exists a continuous semigroup homomorphism $x : \mathbb{R}_+ \to G$ such that

$$\mu_t = \varepsilon_{x(t)}$$

for all $t \geqslant 0$.

One observes that $(\mu_t)_{t \geqslant 0} \in \mathrm{CS}(G)$.

Extending x to a continuous group homomorphism $\varphi : \mathbb{R} \to G$ by

$$\varphi(s) := \begin{cases} x(s) & \text{if } s \geqslant 0 \\ -x(s) & \text{if } s < 0 \end{cases}$$

and looking at the dual homomorphism $h := \varphi^\wedge : G^\wedge \to \mathbb{R}$ one obtains that

$$\hat{\varepsilon}_{x(t)} = \exp(-ith)$$

for all $t \geqslant 0$.

Conversely, given a continuous homomorphism $h : G^\wedge \to \mathbb{R}$ and introducing

$$x := \mathrm{Res}_{\mathbb{R}_+} h^\wedge : \mathbb{R}_+ \to G^{\wedge\wedge} \cong G$$

the translation semigroup $(\varepsilon_{x(t)})_{t \geqslant 0} \in \mathrm{CS}(G)$ satisfies the above Fourier representation. Here Pontryagin's theorem 4.2.13 has been applied.

In summing up we have established a one-to-one correspondence between the sets of translation semigroups $(\varepsilon_{x(t)})_{t \geqslant 0}$ in $M_+^{(1)}(G)$ and of continuous homomorphisms $h : G^\wedge \to \mathbb{R}$ such that

$$(\varepsilon_{x(t)})_{t \geqslant 0} \leftrightarrow \psi \,,$$

where $\psi := ih$.

5.2.7.2 For $\mu \in M_+^{\mathrm{b}}(G)$ with $\|\mu\| \leqslant \alpha$ one introduces the convolution semigroup $(\mu_t)_{t \geqslant 0}$ in $M_+^{(1)}(G)$ by

$$\mu_t := \mathrm{e}^{-t\alpha} \exp(t\mu)$$

with

$$\exp(t\mu) := \sum_{n \geqslant 0} \frac{(t\mu)^n}{n!}$$

for all $t \geqslant 0$, where $\mu^0 := \varepsilon_0$. Obviously

$$\hat{\mu}_t = \exp(-t(\alpha - \hat{\mu}))$$

with $\alpha - \hat{\mu} \in \mathrm{CND}(G^{\wedge})$, hence

$$(\mu_t)_{t \geqslant 0} \leftrightarrow \alpha - \hat{\mu} \, .$$

For $\alpha := 1$ $(\mu_t)_{t \geqslant 0}$ is said to be the **Poisson semigroup determined by** ψ.

In order to stress the determining measure μ we shall write $(\mathrm{e}(t\mu))_{t \geqslant 0}$ instead of $(\mu_t)_{t \geqslant 0}$. But then $(\mathrm{e}(t\varepsilon_{x_0}))_{t \geqslant 0}$ appears to be the **Poisson semigroup with parameter** $x_0 \in G$.

Let $(\mu_t)_{t>0}$ be a semigroup in $\mathrm{S}(G)$. For every $t > 0$ we shall examine the set

$$H_t := \{\chi \in G^{\wedge} : \hat{\mu}_t(\chi) \neq 0\} \, .$$

Properties 5.2.8

5.2.8.1 H_t *is independent of* $t > 0$ *and will therefore be denoted by* H.

5.2.8.2 H *is an open subgroup of* $G^{\wedge}$.

5.2.8.3 $K := H^{\perp} = \mathrm{Inv}(\mu_t)$ *for all* $t > 0$, *and* K *will be called the* **invariance group** *of the semigroup* $(\mu_t)_{t>0}$.

5.2.8.4 *If* $(\mu_t)_{t>0}$ *is symmetric then*

$$\tau_{\mathrm{w}} - \lim_{t \to 0} \mu_t = \omega_K \, ,$$

hence $(\mu_t)_{t \geqslant 0} \in \mathrm{CS}(G)$.

More generally,

5.2.8.5 $(\mu_t)_{t>0} \in \mathrm{S}(G)$ *is* τ_w*-continuous if and only if the limit relationship of Property 5.2.8.4 holds.*

5.2.8.6 $(\mu_t)_{t>0} \in \mathrm{S}(G)$ *is a convolution semigroup* $(\in \mathrm{CS}(G))$ *if and only if* $(\mu_t)_{t>0}$ *is* τ_{w}*-continuous and has invariance group* $K = \{0\}$.

It suffices to provide arguments for Properties 5.2.8.1 to 5.2.8.4. Concerning 5.2.8.1 we note that from the homomorphism property of $t \mapsto \mu_t$ we obtain that

$$H_s \subset H_t = H_{nt} \subset H_s$$

whenever $0 < t < s$ and $n \in \mathbb{N}$ is chosen such that $s < nt$. The assertion follows. For 5.2.8.2 we observe that H is a subgroup of G if and only if $\mathbb{1}_H \in \mathrm{PD}(G)$. In order to show this characterizing property we assume without loss of generality that $(\mu_t)_{t>0}$ is symmetric; the general case follows via symmetrization (see 5.2.6.1) by just looking at the equality

$$H_t = \{\chi \in G^{\wedge} : (\mu_t * \widetilde{\mu_t})^{\wedge}(\chi) \neq 0\}$$

for $t > 0$. For symmetric $(\mu_t)_{t>0}$, however, we have $0 \leqslant \hat{\mu}_t \leqslant 1$, hence

$$\tau_{\mathrm{p}}\text{--}\lim_{n\to\infty} \hat{\mu}_{\frac{1}{n}} = \lim_{n\to\infty} \hat{\mu}_1^n = \mathbb{1}_H \,,$$

and from Property 4.3.3.4 we conclude that $\mathbb{1}_H \in \mathrm{PD}(G^\wedge)$. Since $\hat{\mu}_1 > 0$ in a neighbourhood of 0, H is open (hence closed), thus $\mathbb{1}_H \in \mathrm{CPD}(G^\wedge)$.

For a proof of 5.2.8.3 one just refers to the fact that $\mathrm{supp}(\hat{\mu}_t) = H$ and applies Property 4.4.18.1.

Finally we provide an argument for 5.2.8.4. Since the function $t \mapsto \hat{\mu}_t(\chi)$ is decreasing for each $\chi \in G^\wedge$ and consequently

$$\tau_{\mathrm{p}}\text{--}\lim_{t\to 0} \hat{\mu}_t = \mathbb{1}_H$$

holds (by the above limit relations), the continuity theorem 4.3.8 applies, and together with 4.4.20 it implies that

$$\tau_{\mathrm{w}}\text{--}\lim_{t\to 0} \mu_t = \omega_K \,.$$

In the subsequent discussion we shall employ the symbol $\mathrm{S}(G, K)$ for the set of all τ_{w}-continuous semigroups in $\mathrm{S}(G)$ admitting K as invariance group. Clearly, $\mathrm{S}(G, \{0\}) = \mathrm{CS}(G)$.

Theorem 5.2.9 (Generalized Schoenberg correspondence)
There is a one-to-one correspondence

$$(\mu_t)_{t>0} \leftrightarrow \psi$$

between the set $\mathrm{S}(G, K)$ and the set $\mathrm{CND}(H)$ of continuous negative definite functions ψ on the open subgroup $H := K^\perp$ of $G^\wedge$ given by

$$\hat{\mu}_t(\chi) = \begin{cases} \exp(-t\psi(\chi)) & \text{if } \chi \in H \\ 0 & \text{if } \chi \notin H. \end{cases}$$

Remark 5.2.10 First of all we realize that by 4.2.17.5 $K^\perp$ is indeed an open subgroup of $G^\wedge$.

Moreover, given any open subgroup H of $G^\wedge$ and $\psi \in \mathrm{CND}(H)$ the τ_{w}-continuous semigroup $(\mu_t)_{t>0} \in \mathrm{S}(G)$ with

$$(\mu_t)_{t>0} \leftrightarrow \psi$$

has $K := H^\perp$ as its invariance group, hence belongs to $\mathrm{S}(G, K)$.

The **proof of Theorem 5.2.9** is performed similar to that of Theorem 5.2.3. One just has to observe that in constructing the semigroup $(\mu_t)_{t>0} \in \mathrm{S}(G)$ the functions appearing in the Fourier representation of $(\mu_t)_{t>0}$

belong to $\mathrm{CND}(G^\wedge)$. But this follows from Property 4.3.2.5, since H is an open subgroup of $G^\wedge$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 5.2.11 For the special choice $K = \{0\}$, i.e. $H = G^\wedge$, the correspondence theorems 5.2.3 and 5.2.9 coincide.

Until now we studied convolution semigroups corresponding to continuous negative definite functions. A further object to associate with them is the resolvent family originating from the potential theory of semigroups of operators.

Definition 5.2.12 A (one-parameter) family $(\varrho_\lambda)_{\lambda>0}$ of measures in $M^{\mathrm{b}}_+(G)$ with $\|\lambda\varrho_\lambda\| \leqslant 1$ for all $\lambda > 0$ is called a **resolvent** (of measures) in $M^{\mathrm{b}}_+(G)$ if it satisfies the resolvent equation

$$\varrho_\lambda - \varrho_\mu = (\mu - \lambda)\varrho_\lambda * \varrho_\mu$$

valid for all $\lambda, \mu > 0$.

In analogy to the invariance group of a semigroup of measures we introduce for a resolvent of measures $(\varrho_\lambda)_{\lambda>0}$ in $M^{\mathrm{b}}_+(G)$ the set

$$L_\lambda := \{\chi \in G^\wedge : \hat{\varrho}_\lambda(\chi) \neq 0\}$$

for $\lambda > 0$ and show the following

Properties 5.2.13

5.2.13.1 L_λ *is independent of* $\lambda > 0$ *and will therefore be denoted by* L.

5.2.13.2 L *is an open subgroup of* $G^\wedge$.

5.2.13.3 $M := L^\perp = \mathrm{Inv}(\varrho_\lambda)$ *for all* $\lambda > 0$, *and* M *will be called the* ***invariance group*** *of the resolvent* $(\varrho_\lambda)_{\lambda>0}$.

5.2.13.4

$$\tau_{\mathrm{w}} - \lim_{\lambda\to\infty} \lambda\varrho_\lambda = \omega_M .$$

For the proofs of these properties we simulate the arguments provided for the Properties 5.2.8. It suffices to verify that $\mathbb{1}_L \in \mathrm{CPD}(G^\wedge)$.

In fact

$$\tau_{\mathrm{p}} - \lim_{\lambda\to\infty} \lambda\hat{\varrho}_\lambda = \mathbb{1}_L ,$$

since

$$\hat{\varrho}_\lambda = \frac{\hat{\varrho}_1}{1 + (\lambda - 1)\hat{\varrho}_1}$$

for all $\lambda > 0$ by the resolvent equation, and this implies

$$\lim_{n \to \infty} \lambda \, \hat{\varrho}_\lambda(\chi) = 1$$

whenever $\chi \in L$. Property 4.3.3.4 together with the openness of L yields the assertion.

In what follows we shall use the symbol $\mathrm{R}(G, M)$ for the set of all resolvents in $M_+^{\mathrm{b}}(G)$ admitting M as their invariance group. An analogue of the generalized Schoenberg correspondence theorem reads as follows

Theorem 5.2.14 (Resolvent correspondence)
There is a one-to-one correspondence

$$(\mu_t)_{t>0} \leftrightarrow (\varrho_\lambda)_{\lambda>0}$$

between the sets $\mathrm{S}(G, K)$ *and* $\mathrm{R}(G, K)$ *given by*

$$\varrho_\lambda = \int_0^\infty \mathrm{e}^{-\lambda t} \mu_t \, \mathrm{d}t \, ,$$

where this equality is understood as an equality of Radon measures on G.

In the case of this correspondence $(\mu_t)_{t>0}$ *and* $(\varrho_\lambda)_{\lambda>0}$ *are said to be* **associated***.*

Proof. 1. Starting with a semigroup $(\mu_t)_{t>0} \in \mathrm{S}(G, K)$ we define Radon measures $\varrho_\lambda \in M_+(G)$ by

$$\int f \, \mathrm{d}\varrho_\lambda := \int_0^\infty \mathrm{e}^{-\lambda t} \left(\int f \, \mathrm{d}\mu_t \right) \mathrm{d}t$$

for all $f \in C^{\mathrm{c}}(G)$. Applying the generalized Schoenberg correspondence theorem 5.2.9 in the sense of

$$(\mu_t)_{t>0} \leftrightarrow \psi$$

with $\psi \in \mathrm{CND}(H)$, where $H := K^\perp$, we obtain that $\|\lambda \varrho_\lambda\| \leqslant 1$, hence that $\varrho_\lambda \in M_+^{\mathrm{b}}(G)$, and that

$$\hat{\varrho}_\lambda = \begin{cases} \frac{1}{\lambda + \psi(\chi)} & \text{if } \chi \in H \\ 0 & \text{if } \chi \notin H. \end{cases}$$

From this Fourier representation it follows that $(\varrho_\lambda)_{\lambda>0}$ is a resolvent in $M_+^{\mathrm{b}}(G)$, and that it determines the function ψ as well as the τ_{w}-continuous semigroup $(\mu_t)_{t>0}$ uniquely. It is clear that $(\varrho_\lambda)_{\lambda>0}$ has the same invariance group as $(\mu_t)_{t>0}$, hence $\in \mathrm{R}(G, K)$.

2. Now, let $\chi \in H := K^{\perp}$. From the resolvent equation we deduce that the number

$$\psi_\lambda(\chi) := \frac{1 - \lambda\, \hat{\varrho}_\lambda(\chi)}{\hat{\varrho}_\lambda(\chi)}$$

is independent of $\lambda > 0$. Consequently $\psi := \psi_\lambda$ is a well-defined complex-valued function on H . In the proof of Properties 5.2.13 we showed that

$$\mathbb{1}_H = \tau_{\mathrm{p}}-\lim_{\lambda\to\infty} \lambda\hat{\varrho}_\lambda\,.$$

On the other hand

$$\lambda\, \hat{\varrho}_\lambda(\chi)\, \psi(\chi) = \lambda\, (1 - \lambda\, \hat{\varrho}_\lambda(\chi))$$

for all $\chi \in H$, $\lambda > 0$, hence

$$\psi(\chi) = \mathbb{1}_H(\chi)\, \psi(\chi) = \lim_{\lambda\to\infty} \lambda\, \hat{\varrho}_\lambda(\chi)\, \psi(\chi) = \lim_{\lambda\to\infty} \lambda\, (1 - \lambda\, \hat{\varrho}_\lambda(\chi))$$

for all $\chi \in H$, and consequently $\psi \in \mathrm{CND}(H)$ by Property 4.3.3.4. But now Theorem 5.2.9 implies the assertion, since

$$\left(\int_0^\infty \mathrm{e}^{-\lambda t} \mu_t\, \mathrm{d}t \right)^{\wedge} = \frac{1}{\lambda + \psi} \mathbb{1}_H = \hat{\varrho}_\lambda\,,$$

whenever $\lambda > 0$.   □

Corollary 5.2.15 *A resolvent $(\varrho_\lambda)_{\lambda>0}$ in $M_+^{\mathrm{b}}(G)$ determines a convolution semigroup $(\mu_t)_{t\geqslant 0}$ in $M_+^{(1)}$ if and only if the invariance group of $(\varrho_\lambda)_{\lambda>0}$ equals $\{0\}$.*

This is immediate from the theorem together with Property 5.2.8.5.

Theorem 5.2.16 (Support correspondence)
Let $(\mu_t)_{t\geqslant 0} \in \mathrm{CS}(G)$, and let $(\varrho_\lambda)_{\lambda>0}$ be a resolvent in $M_+^{\mathrm{b}}(G)$ with

$$(\mu_t)_{t\geqslant 0} \leftrightarrow (\varrho_\lambda)_{\lambda>0}\,.$$

*Then for each $\lambda > 0$ *

(i) $\mathrm{supp}(\varrho_\lambda) = \left(\bigcup_{t>0} \mathrm{supp}(\mu_t) \right)^{-}$,

(ii) $\mathrm{supp}(\varrho_\lambda)$ *is a σ -compact semigroup in G with $0 \in \mathrm{supp}(\varrho_\lambda)$ *,

and

(iii) $S := \left[\bigcup_{t>0} \mathrm{supp}(\mu_t) \right]^{-}$ *is σ -compact.*

Proof. We fix $\lambda > 0$ and $f \in C_+^c(G)$. Then

$$\int f \, \mathrm{d}\varrho_\lambda = 0 \text{ if and only if } \int f \, \mathrm{d}\mu_t = 0 \text{ for all } t > 0.$$

Since by assumption on $(\mu_t)_{t \geqslant 0}$

$$\tau_v - \lim_{t \to 0} \mu_t = \varepsilon_0 \,,$$

$f(0) > 0$ implies $\int f \, \mathrm{d}\varrho_\lambda = 0$, hence $0 \in \mathrm{supp}(\varrho_\lambda)$. Similarly, given an open subset U of G we have

$$\mathrm{supp}(\varrho_\lambda) \subset \complement U \text{ if and only if } \mathrm{supp}(\mu_t) \subset \complement U$$

for all $t > 0$, and

$$\mathrm{supp}(\varrho_\lambda) = \left(\bigcup_{t>0} \mathrm{supp}(\mu_t) \right)^-$$

as asserted in (i).

From the support formula quoted in Theorem 4.1.10 we conclude that for $s, t > 0$

$$\mathrm{supp}(\mu_s) + \mathrm{supp}(\mu_t) \subset \mathrm{supp}(\mu_{t+s}) \,,$$

hence that

$$\left(\bigcup_{t>0} \mathrm{supp}(\mu_t) \right)^-$$

is a semigroup which we have shown to contain 0. Since every measure in $M_+^b(G)$ has σ-compact support, (i) implies (ii).

Finally,

$$S = \left[\bigcup_{t>0} \mathrm{supp}(\mu_t) \right]^- = (\mathrm{supp}(\varrho_\lambda) - \mathrm{supp}(\varrho_\lambda))^-$$

But the set $\mathrm{supp}(\varrho_\lambda) - \mathrm{supp}(\varrho_\lambda)$ being σ-compact, also its closure and hence S is σ-compact. This proves (iii). $\qquad\square$

Theorem 5.2.17 *Let* $(\mu_t)_{t \geqslant 0} \in \mathrm{CS}(G)$ *and* $\psi \in \mathrm{CND}(G^\wedge)$ *with*

$$(\mu_t)_{t \geqslant 0} \leftrightarrow \psi \,,$$

and let $S := \left[\bigcup_{t>0} \mathrm{supp}(\mu_t) \right]^-$.

Then

(i) $T := \mathrm{Inv}(\psi) = \{\chi \in G^\wedge : \psi(\chi) = \psi(0)\}$

and

(ii) $S = T^\perp$, *hence* $T = S^\perp$.

Proof.   For each $t > 0$ we define the set

$$A_t := \{\chi \in G^\wedge : \exp(-t\psi(\chi)) = \exp(-t\psi(0))\}.$$

Clearly,

$$\chi \in \mathrm{Inv}(\psi) \text{ if and only if } \chi \in \mathrm{Inv}(\exp(-t\psi))$$

for all $t > 0$. By Property 4.4.18.2 this implies that

$$\mathrm{Inv}(\psi) = \bigcap_{t>0} A_t = \{\chi \in G^\wedge : \psi(\chi) = \psi(0)\},$$

hence (i).

Moreover, Properties 4.4.18.1 and 4.4.18.2 lead to

$$A_t = \mathrm{Inv}(\hat{\mu}_t) = (\mathrm{supp}(\mu_t))^\perp$$

for all $t > 0$, and therefore to

$$(\mathrm{Inv}(\psi))^\perp = \left(\bigcap_{t>0} (\mathrm{supp}(\mu_t))^\perp\right)^\perp$$

$$= \left(\bigcup_{t>0} \mathrm{supp}(\mu_t)\right)^{\perp\perp}$$

which by Property 4.2.17.1 yields the assertion in (ii).   □

By passing to quotient groups and applying the Pontryagin duality the discussion starting with Properties 4.4.19 can be extended in an obvious way.

Properties 5.2.18

5.2.18.1 *If* $(\mu_t)_{t>0}$ *is a semigroup in* $M_+^{(1)}(G)$ *with invariance group* K, *then* $(\dot{\mu}_t)_{t>0}$ *is a semigroup in* $M_+^{(1)}(G/K)$ *with invariance group* $\{0\}$ *(in* G/K).

If, moreover, $(\mu_t)_{t>0} \in \mathrm{S}(G, K)$, *then* $(\dot{\mu}_t)_{t>0} \in \mathrm{S}(G/K, \{0\}) = \mathrm{CS}(G/K)$.

5.2.18.2 *Let* $(\varrho_\lambda)_{\lambda>0} \in \mathrm{R}(G, K)$ *and let* $(\mu_t)_{t>0} \in \mathrm{S}(G, K)$ *with*

$$(\mu_t)_{t>0} \leftrightarrow (\varrho_\lambda)_{\lambda>0}.$$

Then $(\dot{\varrho}_\lambda)_{\lambda>0}$ is a resolvent in $M_+^{\mathrm{b}}(G/K)$ with

$$(\dot{\varrho}_\lambda)_{\lambda>0} \leftrightarrow (\dot{\mu}_t)_{t>0} \in \mathrm{CS}(G/K)\,.$$

5.2.18.3 Let $(\mu_t)_{t\geqslant 0} \in \mathrm{CS}(G)$ and $\psi \in \mathrm{CND}(G^\wedge)$ with

$$(\mu_t)_{t\geqslant 0} \leftrightarrow \psi\,.$$

Then the quotient function $\dot{\psi} \in \mathrm{CND}(G^\wedge/S)$ corresponds to $(\mu_t)_{t\geqslant 0}$ considered as an element of $\mathrm{CS}(T)$, where $S := \mathrm{Inv}(\psi)$ and $T := S^\perp$.

5.3 Lévy functions

In the previous section it was shown that the constant functions, homomorphisms and nonnegative quadratic forms on the dual $G^\wedge$ of a locally compact Abelian group G are negative definite. The question arises of whether these types of negative definite functions can be viewed as building blocks of general negative definite functions on $G^\wedge$. An answer to this question can be given within different frameworks. In probabilistic terms it covers the canonical decomposition due to P. Lévy and A.I. Khinchin of infinitely divisible or embeddable probability measures on Euclidean space (see Theorem 3.4.20). In the present exposition we shall present a canonical representation of negative definite functions on the dual $G^\wedge$ of a locally compact Abelian group G which by the Schoenberg correspondence provides also a canonical representation of the associated convolution semigroups on G.

We start with the discussion of a fundamental centering method developed in terms of a centering function (or local inner product) for G.

Definition 5.3.1 A function $g \in C(G \times G^\wedge)$ is called a **Lévy function** for G if the following conditions are satisfied

(LF1) For every compact $C \subset G^\wedge$

$$\sup_{x \in G} \sup_{\chi \in C} |g(x, \chi)| < \infty.$$

(LF2)

$$g(x, \chi + \varrho) = g(x, \chi) + g(x, \varrho)$$

and

$$g(-x, \chi) = -g(x, \chi)$$

for all $x \in G$, $\chi, \varrho \in G^\wedge$.

(LF3) For every compact $C \subset G^\wedge$ there exists a $U := U_C \in \mathfrak{V}_G(0)$ such that

$$\chi(x) = \exp(ig(x, \chi))$$

whenever $x \in U$, $\chi \in C$.

(LF4) For every compact $C \subset G^\wedge$

$$\lim_{x \to 0} \sup_{\chi \in C} |g(x, \chi)| = 0.$$

Examples 5.3.2 of Lévy functions.

5.3.2.1 If $G := \mathbb{R}^d$ for $d \geqslant 1$ and if for every $i = 1, \ldots, d$ a function $\zeta_i \in C^b(\mathbb{R})$ is given with the properties that

$$\zeta_i(t) = t$$

for all t from some $U \in \mathfrak{V}_G(0)$ and that

$$\zeta_i(-t) = -\zeta_i(t)$$

for all $t \in \mathbb{R}$, then the function $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$g(x, y) := \sum_{i=1}^{d} \zeta_i(x_i) y_i$$

for all $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ in $\mathbb{R}^d$ is a Lévy function for G.

5.3.2.2 Let $G := \mathbb{T}^d$ for $d \geqslant 1$. Then G can be viewed as the group

$$\{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_i \in \ ] -1, 1] \ \text{ for all } \ i = 1, \ldots, d\}$$

with addition modulo 2. Employing for every $i = 1, \ldots, d$ the function ζ_i introduced in the previous example one obtains a Lévy function g for G by putting

$$g(x, m) := \sum_{i=1}^{d} \zeta_i(x_i) m_i$$

for all $x = (x_1, \ldots, x_d) \in \mathbb{T}^d$ and $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$.

5.3.2.3 Let $G := \mathbb{Q}_d{}^\wedge$ (the dual of the discretely topologized rationals), $\zeta \in C^b(G)$, and let $\chi_0 \in \mathbb{Q}_d$ with $\chi_0 \neq 0$ be such that $\exp(\mathrm{i}\zeta(x)) = \chi_0(x)$ for all x in some $U \in \mathfrak{V}_G(1)$. Then the real-valued function g on $G \times G^\wedge$ defined by

$$g(x, \chi) := \zeta(x) \frac{\chi}{\chi_0}$$

for all $x \in G$, $\chi \in G^\wedge$ is a Lévy function for G.

5.3.2.4 If G is totally disconnected, then the zero function on $G \times G^\wedge$ is a Lévy function for G, since every homomorphism from $G^\wedge$ into $\mathbb{R}$ is trivial.

Properties 5.3.3 of Lévy functions.

5.3.3.1 (Products) *Let* $G = G_1 \times G_2$ *be the product of two locally compact Abelian groups* G_1 *and* G_2 *admitting Lévy functions* g_1 *and* g_2 *respectively. Then* G *admits the Lévy function* g *given by*

$$g(x, \chi) := g_1(x_1, \chi_1) + g_2(x_2, \chi_2)$$

whenever $x := (x_1, x_2) \in G_1 \times G_2$ *and* $\chi := (\chi_1, \chi_2) \in G_1{}^\wedge \times G_2{}^\wedge$.

5.3.3.2 (Extensions) *Let H be an open subgroup of a locally compact Abelian group G admitting a Lévy function g_0. Let $\pi^\wedge$ denote the canonical homomorphism $G^\wedge \to H^\wedge \cong G^\wedge/H^\perp$. Then the function g on $G \times G^\wedge$ defined by*

$$g(x, \chi) := \begin{cases} g_0(x, \pi^\wedge(\chi)) & \text{if } x \in H \\ 0 & \text{if } x \notin H \end{cases}$$

and all $\chi \in G^\wedge$ is a Lévy function for G.

In fact, $g \in C(G \times G^\wedge)$, since H is an open, hence a closed subgroup of G. The remaining properties of g follow from those of g_0.

Theorem 5.3.4 *On any locally compact Abelian group G there exists a Lévy function.*

Proof. 1. From Property 5.3.3.2 we infer that it is sufficient to prove the theorem for an open subgroup H of G. Choosing H as the subgroup generated by an open compact neighbourhood of $0 \in G$ we can apply the structure theorem 4.2.19 and obtain the direct product decomposition

$$H \cong \mathbb{R}^d \times \mathbb{Z}^e \times K$$

where $d, e \geqslant 0$ and K is a compact Abelian group. But now it is obvious by Property 5.3.3.1 that we may without loss of generality assume G to be a compact group, since for $\mathbb{R}^d$ and $\mathbb{Z}^d$ Lévy functions exist by Examples 5.3.2.1 and 5.3.2.4 respectively.

2. Let G be a compact Abelian group. For the connected component G_0 of the identity $0 \in G$ we have

$$G_0{}^\wedge \cong G^\wedge/G_0{}^\perp \quad \text{and} \quad (G/G_0)^\wedge \cong G_0{}^\perp$$

by the functorial properties 4.2.17 of the duality. Since G_0 is connected and compact, $G_0{}^\wedge$ is a discrete group in which each element is of infinite order. An application of Zorn's lemma yields a maximal family $\{d_\alpha : \alpha \in A\}$ in $G_0{}^\wedge$ with the following linear independence property: If

$$\sum_{i=1}^{l} n_{\alpha_i} d_{\alpha_i} = 0$$

for some $l \in \mathbb{N}$, $n_{\alpha_1}, \ldots, n_{\alpha_l} \in \mathbb{Z}$ and $d_{\alpha_1}, \ldots, d_{\alpha_l} \in \{d_\alpha : \alpha \in A\}$, then $n_{\alpha_i} = 0$ for all $i = 1, \ldots, l$. But then for any $d \in G_0{}^\wedge$ there exist $d_{\alpha_1}, \ldots, d_{\alpha_k} \in \{d_\alpha : \alpha \in A\}$ and $n, n_1, \ldots, n_k \in \mathbb{Z}$, $n > 0$ such that

$$nd = \sum_{i=1}^{k} n_i d_{\alpha_i}, \tag{1}$$

where the representation is unique within multiplication by integers. Each element of $G_0{}^{\wedge}$ being a coset of $G_0{}^{\perp}$ in $G^{\wedge}$ we take the coset d_{α} and choose an element $\chi_{\alpha} \in G^{\wedge}$ from this coset. χ_{α} will be fixed for the moment. One observes that there exists a function $h_{\chi_{\alpha}} \in C(G)$ having the following properties:

$$|h_{\chi_{\alpha}}| \leqslant \pi,$$

$$h_{\chi_{\alpha}}(-x) = -h_{\chi_{\alpha}}(x)$$

for all $x \in G$, and

$$\chi_{\alpha}(x) = \exp(ih\chi_{\alpha}(x))$$

for all $x \in G$ such that $|\chi_{\alpha}(x) - 1| \leqslant \frac{1}{2}$. Next we define

$$g(x, \chi_{\alpha}) := h\chi_{\alpha}(x)$$

for all $x \in G$ and $\alpha \in A$.

Now, let $\chi \in G^{\wedge}$ be arbitrary. Then χ belongs to some coset of $G_0{}^{\perp}$ being an element of $G^{\wedge}/G_0{}^{\perp}$. This element d admits a representation of the form (1) with $n, n_1, \ldots, n_k \in \mathbb{Z}$, $n > 0$ and $d_{\alpha_1}, \ldots, d_{\alpha_k} \in \{d_{\alpha} : \alpha \in A\}$ $(k \geqslant 1)$. The function g defined by

$$g(x, \chi) := \sum_{j=1}^{k} \frac{n_j}{n} g(x, \chi_{\alpha_j})$$

for all $x \in G$, $\chi \in G^{\wedge}$ will serve as a Lévy function for G. We shall show the required properties in

3. First of all $g \in C(G \times G^{\wedge})$, since $x \mapsto g(x, \chi)$ is continuous on G for each $\chi \in G^{\wedge}$ and $G^{\wedge}$ is discrete. Properties (LF1) and (LF2) follow directly from the construction.

Since compact sets in $G^{\wedge}$ are finite it suffices to prove Property (LF3) for each $\chi \in G^{\wedge}$. For any $\chi \in G^{\wedge}$ we denote the coset of $G_0{}^{\perp}$ to which χ belongs, by $\dot{\chi}$. Then (1) can be rewritten in the form

$$n\dot{\chi} = \sum_{j=1}^{k} n_j \dot{\chi}_{\alpha_j} . \tag{2}$$

Clearly, $\chi_1 - \chi_2 \in G_0{}^{\perp}$ whenever $\chi_1, \chi_2 \in \dot{\chi}$. Since $G_0{}^{\perp} = (G/G_0)^{\wedge}$ is totally disconnected, every element of $G_0{}^{\perp}$ is of finite order. Hence for any $\chi \in G_0{}^{\perp}$ there exists a neighbourhood $V \in \mathfrak{V}_G(0)$ such that $\chi(x) = 1$ for all $x \in V$, and consequently, for $\chi_1, \chi_2 \in \dot{\chi}$ we have $\chi_1(x) = \chi_2(x)$ for all x in some $V_1 \in \mathfrak{V}_G(0)$.

But now we refer to the construction of the function h_{χ_α} above. There exists a neighbourhood $V_2 \in \mathfrak{V}_G(0)$ such that

$$\chi_\alpha(x) = \exp(ig(x, \chi_\alpha))$$

for all $x \in V_2$ ($\alpha \in A$). For arbitrary $\chi \in G^\wedge$ we employ (2) in order to obtain the representation

$$n\chi = \sum_{j=1}^{k} (\chi_{\alpha_j 1} + \cdots + \chi_{\alpha_j n_j}), \tag{3}$$

where $\chi_{\alpha_j 1}, \ldots, \chi_{\alpha_j n_j} \in \dot{\chi}_{\alpha_j}$ for $j = 1, \ldots, k$. For the above discussion it follows that there exists a neighbourhood $V_3 \in \mathfrak{V}_G(0)$ (depending on the characters $\chi_{\alpha_j r}$ and χ_{α_j}) such that

$$\chi_{\alpha_j r}(x) = \chi_{\alpha_j}(x)$$

for all $x \in V_3$ ($r = 1, \ldots, n_j$, $j = 1, \ldots, k$). Let U_1 denote the intersection of all neighbourhoods of type V_3 arising from the choices of the $\chi_{\alpha_j r}$ ($r = 1, \ldots, n_j$, $j = 1, \ldots, k$). Then

$$\chi_{\alpha_j r}(x) = \chi_{\alpha_j}(x) \tag{4}$$

for all $x \in U_1$, $r = 1, \ldots, n_j$, $j = 1, \ldots, k$. But (3) and (4) imply that

$$\chi(x)^n = \prod_{j=1}^{k} \chi_{\alpha_j}(x)^{n_j}$$

for all $x \in U_1$. Since there are neighbourhoods $\in \mathfrak{V}_G(0)$, on which

$$\chi_{\alpha_j} = \exp(ig(\cdot, \chi_{\alpha_j}))$$

holds, there is also a neighbourhood $\in \mathfrak{V}_G(0)$ where

$$\chi^n = \exp(ing(\cdot, \chi))$$

is valid. χ and $\exp(ig(\cdot, \chi))$ being continuous and $\neq 0$ at the identity 0 of G we obtain the desired neighbourhood $U \in \mathfrak{V}_G(0)$ such that

$$\chi(x) = \exp(ig(x, \chi))$$

for all $x \in U$, $\chi \in G^\wedge$.

Finally we note that Property (LF4) follows from the facts that $g \in C(G \times G^\wedge)$ and that $g(x, \chi) = 0$ whenever either x or χ are the identities of G or $G^\wedge$ respectively. The proof is complete. $\qquad\square$

Applying a Lévy function g for G we can provide a new type of continuous negative definite functions on $G^\wedge$ which will be crucial for the canonical representation we are aiming at

Theorem 5.3.5 *Let* g *be a Lévy function for* G, *and let* $\mu \in M_+(G^\times)$ *with* $G^\times := G \setminus \{0\}$ *be such that*

$$\int_{G^\times} (1 - \mathrm{Re}\,\chi)\,\mathrm{d}\mu < \infty$$

for all $\chi \in G^\wedge$. *Then the function* ψ_μ *on* G *defined by*

$$\psi_\mu(\chi) := \int_{G^\times} \big(1 - \chi(x) + \mathrm{i}g(x,\chi)\big)\,\mu(\mathrm{d}x)$$

for all $\chi \in G^\wedge$ *belongs to* $\mathrm{CND}(G^\wedge)$.

Proof. For a fixed $\chi_0 \in G^\wedge$ and a compact neighbourhood $W \in \mathfrak{V}_{G^\wedge}(0)$ there exists by Property (LF3) of g a neighbourhood $U = U(W) \in \mathfrak{V}_G(0)$ such that

$$\chi(x) = \exp(\mathrm{i}g(x,\chi))$$

for all $x \in U$ and $\chi \in \chi_0 + W$. Then Property (LF4) implies the existence of a neighbourhood $U' \in \mathfrak{V}_G(0)$ satisfying the inequality

$$|\sin g(x,\chi) - g(x,\chi)| \leqslant 1 - \cos g(x,\chi)$$

valid for all $x \in U'$ and $\chi \in \chi_0 + W$. It follows that for all $x \in U'$ and $\chi \in \chi_0 + W$

$$\big|1 - \chi(x) + \mathrm{i}\,g(x,\chi)\big| = |1 - \cos g(x,\chi) - \mathrm{i}\sin g(x,\chi) + \mathrm{i}\,g(x,\chi)|$$
$$\leqslant 2|1 - \mathrm{Re}\,\chi(x)| \tag{5}$$

Since $\mu(G^\times \setminus U') < \infty$ by assumption, Property (LF1) implies that $\psi_\mu(\chi_0)$ is a well-defined number $\in \mathbb{C}$ and moreover that there exists a constant $c := c(\chi_0) \geqslant 0$ such that

$$|\psi_\mu(\chi)| \leqslant 2\,\mathrm{Re}\,\psi_\mu(\chi) + c(\chi_0) \tag{6}$$

whenever $\chi \in \chi_0 + W$. Since for every $x \in G$ the functions

$$\chi \mapsto 1 - \chi(x) + \mathrm{i}g(x,\chi)$$

and

$$\chi \mapsto 1 - \mathrm{Re}\,\chi(x)$$

belong to $\mathrm{ND}(G^\wedge)$, also ψ_μ and $\mathrm{Re}\,\psi_\mu$ are elements of $\mathrm{ND}(G^\wedge)$. Here Example 5.1.8.2 and Properties 5.1.3 applied.

It remains to show that ψ_μ is continuous. Once we know that $\operatorname{Re}\psi_\mu$ is locally bounded, inequality (6) justifies the local boundedness of ψ_μ. Therefore let ψ_μ be of the form

$$\chi \mapsto \int_{G^\times} (1 - \operatorname{Re}\chi)\,\mathrm{d}\mu\,.$$

Let $\mathcal{K}$ denote the family of the compact subsets K of G with $0 \notin K$, and set for each $K \in \mathcal{K}$

$$\mu_K := \operatorname{Res}_K \mu\,.$$

Clearly $\psi_{\mu_K} \in \operatorname{CND}(G^\wedge)$ for all $K \in \mathcal{K}$, hence

$$\psi_\mu := \sup_{K \in \mathcal{K}} \psi_{\mu_K} \in \operatorname{ND}(G^\wedge)$$

is lower semicontinuous. For every $n \geqslant 1$ the set $A_n := \{\chi \in G^\wedge : \psi_\mu(\chi) \leqslant n\}$ is closed. Since ψ_μ is finite everywhere we have $G^\wedge = \bigcup_{n \geqslant 1} A_n$, and by Baire's theorem there exists an $n_0 \geqslant 1$ such that $A_{n_0} \neq \emptyset$. Let V be an open neighbourhood $\in \mathfrak{V}_G(0)$ and let $\chi_1 \in A_{n_0}$ be such that $\chi_1 + V \subset A_{n_0}$. Since $\psi_\mu \in \operatorname{ND}(G^\wedge)$ we obtain from Property 5.1.2.3 for all $\chi \in \chi_0 + V$ $(\chi_0 \in G^\wedge)$ and $\chi_2 := \chi_0 - \chi_1$ that

$$\begin{aligned}
\sqrt{\psi_\mu(\chi)} &= \sqrt{\psi_\mu(\chi - \chi_2 + \chi_2)} \\
&\leqslant \sqrt{\psi_\mu(\chi - \chi_2)} + \sqrt{\psi_\mu(\chi_2)} \\
&\leqslant \sqrt{\psi_\mu(\chi_2)} + \sqrt{n_0}\,,
\end{aligned}$$

hence that ψ_μ is locally bounded. Since ψ_μ is a complex linear combination of linear semicontinuous functions, it is locally $\omega_{G^\wedge}$-integrable, and given a function $f \in C_+^c(G^\wedge)$ satisfying $f(\chi) = f(-\chi)$ for all $\chi \in G^\wedge$ and $\int f\,\mathrm{d}\omega_{G^\wedge} = 1$ we conclude that

$$\begin{aligned}
\psi_\mu * f(\chi) &= \int_{G^\wedge} f(\chi - \varrho) \left(\int_{G^\times} \big(1 - \varrho(x) + \mathrm{i}g(x,\varrho)\big)\,\mu(\mathrm{d}x) \right) \omega_{G^\wedge}(\mathrm{d}\varrho) \\
&= \int_{G^\times} \big(1 - \chi(x)\,\mathcal{F}_{G^\wedge}f(x) + \mathrm{i}g(x,\chi)\big)\,\mu(\mathrm{d}x) \\
&= \psi_\mu(\chi) + \int_{G^\times} \chi(x)(1 - \mathcal{F}_{G^\wedge}f(x))\,\mu(\mathrm{d}x)
\end{aligned}$$

whenever $\chi \in G^\wedge$. But this implies the continuity of ψ_μ, since the last integral in the above chain of equalities is the Fourier transform of the measure $(1 - \mathcal{F}_{G^\wedge}f) \cdot \mu \in M_+^{\mathrm{b}}(G^\times)$ considered as a measure on G. $\qquad\square$

5.4   The Lévy–Khinchin representation

For any locally compact space E we denote by $M^{\mathrm{c}}(E)$ the set of Radon measures with compact support. In the case of a locally compact Abelian group G with dual $G^{\wedge}$ the set
$$S := \{\sigma \in M^1(G^{\wedge}) \cap M^{\mathrm{c}}(G^{\wedge}) : \sigma = \sigma^{\sim}\}$$
of symmetric probability measures on $G^{\wedge}$ with compact support will be efficiently applied.

Theorem 5.4.1 *Let* $\psi \in \mathrm{CND}(G^{\wedge})$.   *Then*

(i) *for any* $\sigma \in S$ *the function* $\psi * \sigma - \psi$ *belongs to* $\mathrm{CPD}(G^{\wedge})$ *and hence admits a Bochner measure* $\mu_{\sigma} \in M_+^{\mathrm{b}}(G)$ *in the sense that*
$$\mathcal{F}_G \mu_{\sigma} = \psi * \sigma - \psi \,.$$

(ii) *There exists a measure* $\mu \in M_+(G^{\times})$ *satisfying* $(1 - \mathcal{F}_{G^{\wedge}}\sigma) \cdot \mu = \mathrm{Res}_{G^{\times}} \mu_{\sigma}$ *whenever* $\sigma \in S$.

(iii) *If* ψ *is associated with a convolution semigroup* $(\mu_t)_{t \geqslant 0}$ *in* $M_+^{(1)}(G)$ *according to the Schoenberg correspondence, then*
$$\mu = \tau_{\mathrm{v}} - \lim_{t \to 0} \frac{1}{t} \, \mathrm{Res}_{G^{\times}} \mu_t \,.$$

Remark 5.4.2 The relevant references to statements (i) and (iii) of the theorem are Theorems 4.3.5 and 5.2.3 respectively.

Definition 5.4.3 The measure μ established in Theorem 5.4.1 is called the **Lévy measure** of the continuous negative definite function ψ or of the convolution semigroup $(\mu_t)_{t \geqslant 0}$ associated with ψ.

Proof.   (Proof of Theorem 5.4.1). Let $\psi \leftrightarrow (\mu_t)_{t \geqslant 0}$ (via Theorem 5.2.3) and let $\sigma \in S$. Then for any $t > 0$
$$(1 - \mathcal{F}_{G^{\wedge}}\sigma) \cdot \frac{1}{t}\mu_t \in M_+^{\mathrm{b}}(G)$$
and
$$\begin{aligned}
\mathcal{F}_G \left[(1 - \mathcal{F}_{G^{\wedge}}\sigma) \cdot \frac{1}{t}\mu_t \right](\chi) &= \frac{1}{t} \int \overline{\chi(x)}(1 - \mathcal{F}_{G^{\wedge}}\sigma(x)) \, \mu_t(\mathrm{d}x) \\
&= \frac{1}{t} \int \overline{\chi(x)} \left(1 - \int \varrho(x)\sigma(\mathrm{d}\varrho) \right) \mu_t(\mathrm{d}x) \\
&= \frac{1}{t} \left(\hat{\mu}_t(\chi) - \iint \overline{(\chi - \varrho)(x)} \, \sigma(\mathrm{d}\varrho) \, \mu_t(\mathrm{d}x) \right) \\
&= \frac{1}{t} (\hat{\mu}_t(\chi) - \hat{\mu}_t * \sigma(\chi)) \\
&= \frac{1}{t} (1 - \exp(-t\psi) * (\sigma - \varepsilon_0))(\chi)
\end{aligned}$$

for all $\chi \in G^\wedge$. Since

$$\tau_{\mathrm{co}}-\lim_{t\to 0}\frac{1}{t}(1-\exp(-t\varphi)) = \psi\,,$$

we obtain that

$$\lim_{t\to 0}\mathcal{F}_G\left[(1-\mathcal{F}_{G^\wedge}\sigma)\cdot\frac{1}{t}\mu_t\right] = \psi * \sigma - \psi$$

in either of the topologies τ_{p} and τ_{co}. As a conclusion we obtain that $\psi * \sigma - \psi \in \mathrm{CPD}(G^\wedge)$, and by an application of Theorem 4.3.7 that

$$\tau_{\mathrm{w}}-\lim_{t\to 0}(1-\mathcal{F}_{G^\wedge}\sigma)\cdot\frac{1}{t}\mu_t = \mu_\sigma\,,$$

where $\mu_\sigma \in M_+^{\mathrm{b}}(G)$ such that

$$\hat{\mu}_\sigma = \psi * \sigma - \psi\,.$$

Given $\varphi \in C_+^{\mathrm{c}}(G)$ with $\mathrm{supp}(\varphi) \subset G^\times$ we choose $\sigma \in S$ such that $\mathcal{F}_{G^\wedge}\sigma(x) \leqslant \frac{1}{2}$ for all x in a neighborhood of $\mathrm{supp}(\varphi)$. The function φ' on G defined by

$$\varphi'(x) := \begin{cases} \frac{\varphi(x)}{1-\mathcal{F}_{G^\wedge}\sigma(x)} & \text{if } x \in \mathrm{supp}(\varphi) \\ 0 & \text{otherwise} \end{cases}$$

belongs to $C_+^{\mathrm{c}}(G)$, and

$$\lim_{t\to 0}\int \varphi\,\mathrm{d}\left(\frac{1}{t}\mu_t\right) = \lim_{t\to 0}\int \varphi'\,\mathrm{d}\left((1-\mathcal{F}_{G^\wedge}\sigma)\cdot\frac{1}{t}\mu_t\right)$$

$$= \int \varphi'\,\mathrm{d}\mu_\sigma\,.$$

This implies that there exists a measure $\mu \in M_+(G^\times)$ such that

$$\mu = \tau_{\mathrm{v}}-\lim_{t\to 0}\frac{1}{t}\,\mathrm{Res}_{G^\times}\mu_t$$

and that

$$(1-\mathcal{F}_{G^\wedge}\sigma)\cdot\mu = \mathrm{Res}_{G^\times}\mu_\sigma$$

holds for all $\sigma \in S$. □

Properties 5.4.4 of Lévy measures.

5.4.4.1 *The Lévy measure μ of a function $\psi \in \mathrm{CND}(G^\wedge)$ (or of its associated convolution semigroup in $M_+^{(1)}(G)$) has the following properties:*

(a)
$$\int_{G^\times} (1 - \operatorname{Re}\chi)\, \mathrm{d}\mu < \infty$$

for all $\chi \in G^\wedge$.

(b) *For every compact neighbourhood* $V \in \mathfrak{V}_G(0)$
$$\operatorname{Res}_{\complement V} \mu \in M_+^{\mathrm{b}}(G)\,.$$

While the statement in (a) follows directly from Theorem 5.4.1 by special-izing σ to the measure $\frac{1}{2}(\varepsilon_\chi + \varepsilon_{-\chi})$ for $\chi \in G^\wedge$, the statement in (b) makes use of the existence of $\sigma \in S$ satisfying
$$1 - \mathcal{F}_{G^\wedge}\sigma(x) \geqslant \frac{1}{2}$$

for all x in the complement $\complement V$ of a compact neighbourhood $V \in \mathfrak{V}_G(0)$; one just applies (ii) of Theorem 5.4.1 and deduces the inequalities
$$\int \varphi\, \mathrm{d}\mu \leqslant 2 \int \varphi\, \mathrm{d}[(1 - \mathcal{F}_{G^\wedge}\sigma) \cdot \mu]$$
$$= 2 \int \varphi\, \mathrm{d}\mu_\sigma$$

valid for each $\varphi \in C_+^{\mathrm{c}}(G)$ with $\operatorname{supp}(\varphi) \subset \complement V$.

5.4.4.2 *Let* $\psi_\mu \in \mathrm{CND}(G^\wedge)$ *be the function introduced for a given measure* $\mu \in M_+(G^\times)$ *in Theorem 5.3.5. The Lévy measure of* ψ_μ *is just* μ, *hence* μ *is uniquely determined by* ψ_μ.

In fact, for $\sigma \in S$ and $\chi \in G^\wedge$ we obtain the identity
$$\psi_\mu * \sigma(\chi) - \psi_\mu(\chi) = \int_{G^\times} \chi(x)(1 - \mathcal{F}_{G^\wedge}\sigma)(x)\, \mu(\mathrm{d}x)\,,$$

and it follows that the measure $(1 - \mathcal{F}_{G^\wedge}\sigma) \cdot \mu$ considered as an element of $M_+^{\mathrm{b}}(G)$ has $\psi_\mu * \sigma - \psi_\mu$ as its Fourier transform.

5.4.4.3 *Let* $\psi_1, \psi_2 \in \mathrm{CND}(G^\wedge)$ *(associated with convolution semigroups* $(\mu_t^{(1)})_{t\geqslant 0}$ *and* $(\mu_t^{(2)})_{t\geqslant 0}$ *respectively) admit* μ_1 *and* μ_2 *as their Lévy measures. Then the function* $\psi := \psi_1 + \psi_2 \in \mathrm{CND}(G^\wedge)$ *(associated with the convolution semigroups* $(\mu_t^{(1)} * \mu_t^{(2)})_{t\geqslant 0}$*) admits* $\mu := \mu_1 + \mu_2$ *as its Lévy measure.*

Example 5.4.5 Let $\mu \in M_+^{\mathrm{b}}(G)$ with $\|\mu\| \leqslant 1$. Then by Example 5.2.7.2 $\psi := 1 - \hat{\mu}$ is the continuous negative definite function associated with the convolution semigroup determined by μ. It turns out that the Lévy measure of ψ is $\operatorname{Res}_{G^\times}\mu$.

By a simple computation of Fourier transforms and an application of their sequential bicontinuity (Theorem 4.3.8) one shows that any convolution semigroup $(\mu_t)_{t \geqslant 0}$ in $M_+^{(1)}(G)$ with associated resolvent $(\varrho_\lambda)_{\lambda > 0}$ can be obtained as a limit of convolution semigroups $(\mu_t^\lambda)_{t \geqslant 0}$ determined by $\lambda^2 \varrho_\lambda$ in the sense that

$$\mu_t = \tau_{\mathrm{w}} - \lim_{\lambda \to \infty} \mu_t^\lambda$$

for all $t > 0$. It is easy to see that for the Lévy measure μ of $(\mu_t)_{t \geqslant 0}$ one obtains

$$\mu = \tau_{\mathrm{v}} - \lim_{\lambda \to \infty} \mathrm{Res}_{G^\times} \lambda^2 \varrho_\lambda \,.$$

We are now prepared to *characterize the two types of continuous negative definite functions* introduced in Examples 5.1.8.2 and 5.1.8.3 in terms of measures in S.

Properties 5.4.6

5.4.6.1 *A real-valued function $h \in C(G^\wedge)$ with $h(0) = 0$ is a (group) homomorphism if and only if $h * \sigma - h = 0$ for all $\sigma \in S$.*

In fact, for any homomorphism $h : G^\wedge \to \mathbb{R}$ and each $\sigma \in S$ we have

$$\int h(\varrho)\,\sigma(\mathrm{d}\varrho) = \int h(-\varrho)\,\sigma^\sim(\mathrm{d}\varrho)$$

$$= -\int h(\varrho)\,\sigma(\mathrm{d}\varrho)\,,$$

hence $\int h \, \mathrm{d}\sigma = 0$ and consequently

$$h * \sigma(\chi) = \int h(\chi - \varrho)\,\sigma(\mathrm{d}\varrho)$$

$$= h(\chi) - \int h(\varrho)\,\sigma(\mathrm{d}\varrho)$$

$$= h(\chi)$$

whenever $\chi \in G^\wedge$. If conversely this latter condition holds for $\sigma \in S$, then we obtain for $\chi, \varrho \in G^\wedge$ and $\sigma := \frac{1}{2}(\varepsilon_\chi + \varepsilon_{-\chi})$ that

$$0 = h * \sigma(\varrho) - h(\varrho) = \frac{1}{2}[h(\varrho - \chi) + h(\varrho + \chi)] - h(\varrho)\,,$$

in particular for the choice $\varrho = 0$ that

$$\frac{1}{2}[h(-\chi) + h(\chi)] = 0\,.$$

Analogously we obtain

$$0 = \frac{1}{2}[h(\chi - \varrho) + h(\chi + \varrho)] - h(\chi)\,,$$

hence adding the two equalities yields

$$h(\chi + \varrho) = h(\chi) + h(\varrho)\,.$$

As an immediate consequence of this property we note that

5.4.6.2 *the Lévy measure of a function* $\psi \in \mathrm{CND}(G^\wedge)$ *of the form* $\psi := \mathrm{i}h$ *with a continuous homomorphism* $h : G^\wedge \to \mathbb{R}$ *is the zero measure.*

With similar arguments one proves that

5.4.6.3 *for a function* $\psi \in \mathrm{CND}(G^\wedge)$ *with Lévy measure* μ *the function* $\mathrm{i}\,\mathrm{Im}\,\psi \in \mathrm{CND}(G^\wedge)$ *if and only if* μ *is symmetric.*

5.4.6.4 *A symmetric real-valued function* $q \in C(G^\wedge)$ *with* $q(0) = 0$ *is a quadratic form on* $G^\wedge$ *if and only if there exists a constant* $c \in \mathbb{R}$ *such that*

$$q * \sigma - q = c$$

whenever $\sigma \in S$. *In the affirmative case* $q \geqslant 0$ *if and only if*

$$q * \sigma - q \geqslant 0$$

for all $\sigma \in S$.

For the proof let q be a quadratic form on $G^\wedge$. Then for $\sigma \in S$ and $\chi \in G^\wedge$ we have

$$\begin{aligned}
q * \sigma(\chi) &= \int q(\chi - \varrho)\,\sigma(\mathrm{d}\varrho) \\
&= \int q(\chi + \varrho)\,\sigma(\mathrm{d}\varrho) \\
&= \int \frac{1}{2}[q(\chi - \varrho) + q(\chi + \varrho)]\,\sigma(\mathrm{d}\varrho) \\
&= \int [q(\chi) + q(\varrho)]\,\sigma(\mathrm{d}\varrho) \\
&= q(\chi) + \int q(\varrho)\,\sigma(\mathrm{d}\varrho)\,.
\end{aligned}$$

If conversely $q * \sigma - q$ equals to a constant $c \in \mathbb{R}$ for all $\sigma \in S$, then for $\chi, \varrho \in G^\wedge$ and $\sigma := \frac{1}{2}(\varepsilon_\varrho + \varepsilon_{-\varrho})$ we obtain

$$q * \sigma(\chi) - q(\chi) = q * \sigma(0) - q(0)$$

or

$$\frac{1}{2}[q(\chi + \varrho) + q(\chi - \varrho)] - q(\chi) = \frac{1}{2}[q(\varrho) + q(-\varrho)] = q(\varrho)\,.$$

These equalities also yield the remaining part of the assertion.

As an immediate consequence of 5.4.6.4 we arrive at the assertion that

5.4.6.5   *a real-valued function $\psi \in \mathrm{CND}(G^\wedge)$ with $\psi(0) = 0$ having Lévy measure μ is a quadratic form $\geqslant 0$ if and only if μ is the zero measure.*

In fact, from 5.4.6.4 follows that ψ is a quadratic form if and only if the measure μ_σ introduced in (i) of Theorem 5.4.1 is supported by $\{0\}$ for all $\sigma \in S$. But this statement is equivalent to μ being the zero measure.

Theorem 5.4.7 (Canonical representation of continuous negative definite functions)

Let g be a Lévy function for G. For any complex-valued function ψ on G the following statements are equivalent:

(i) *$\psi \in \mathrm{CND}(G^\wedge)$.*

(ii) *There exist a constant $c \geqslant 0$, a continuous homomorphism $h : G^\wedge \to \mathbb{R}$, a non-negative continuous quadratic form on $G^\wedge$ and a measure $\mu \in M_+(G^\times)$ satisfying*

$$\int_{G^\times} (1 - \operatorname{Re}\chi)\,\mathrm{d}\mu < \infty$$

for all $\chi \in G^\wedge$ such that

$$\psi(\chi) = c + \mathrm{i}h(\chi) + q(\chi) + \int_{G^\times} \big(1 - \chi(x) + \mathrm{i}g(x,\chi)\big)\,\mu(\mathrm{d}x) \quad (7)$$

whenever $\chi \in G^\wedge$.

*Moreover, in the **canonical quadruple** (c, h, q, μ), the objects $c := \psi(0)$, q given by*

$$q(\chi) = \lim_{n \to \infty} \frac{1}{n^2} \operatorname{Re}\psi(n\chi) \quad (8)$$

for all $\chi \in G^\wedge$, and μ being the Lévy measure of ψ are uniquely determined by ψ, while the function $\mathrm{i}h$ depends on the choice of the Lévy function g.

Proof. (i) $\Rightarrow$ (ii). Let $\psi \in \mathrm{CND}(G^\wedge)$. The Lévy measure μ of ψ coincides with the Lévy measure of $\psi' := \psi - c$. Introducing the function

$$\psi'' := \psi' - \psi_\mu\,,$$

where ψ_μ has been defined in Theorem 5.3.5, we obtain that

$$\psi'' * \sigma - \psi'' = \mu_\sigma(\{0\}), \quad (9)$$

where $\mu_\sigma \in M_+^b(G)$ for $\sigma \in S$ has Fourier transform

$$\mathcal{F}_G(\mu_\sigma) = \psi * \sigma - \psi$$

(see the proof of Property 5.4.4.2). In fact,

$$\begin{aligned}
\psi'' * \sigma - \psi'' &= \psi' * \sigma - \psi' - (\psi_\mu * \sigma - \psi_\mu) \\
&= \mathcal{F}_G(\mu_\sigma(\{0\})\varepsilon_0 + \mathrm{Res}_{G^\times} \mu_\sigma) - \mathcal{F}_G((1 - \mathcal{F}_{G^\wedge}\sigma) \cdot \mu) \\
&= \mu_\sigma(\{0\}).
\end{aligned}$$

Now taking real and imaginary parts in the identity (9) we arrive at the equalities

$$(\mathrm{Re}\,\psi'') * \sigma - \mathrm{Re}\,\psi'' = \mu_\sigma(\{0\})$$

and

$$(\mathrm{Im}\,\psi'') * \sigma - \mathrm{Im}\,\psi'' = 0$$

for all $\sigma \in S$. Since ψ'' is a difference between negative definite functions which are equal to 0 at 0, we have that

$$\mathrm{Im}\,\psi''(0) = \mathrm{Re}\,\psi''(0) = 0$$

and

$$\mathrm{Re}\,\psi''(\chi) = \mathrm{Re}\,\psi''(-\chi)$$

for all $\chi \in G^\wedge$. Application of the characterizations 5.4.6.1 and 5.4.6.4 provides the desired homomorphism $h = \mathrm{Im}\,\psi''$ and the non-negative quadratic form $q := \mathrm{Res}\,\psi''$. The representation (7) has been established.

(ii) $\Rightarrow$ (i). We are now given the function ψ represented as in (7) via the quadruple (c, h, q, μ). Then Example 5.1.8.4 and Theorem 5.3.5 imply that $\psi \in \mathrm{CND}(G^\wedge)$. But by Property 5.4.4.2 together with characterizations 5.4.6.1 and 5.4.6.5 μ is the Lévy measure of ψ and $c = \psi(0)$. It remains to be shown that q satisfies the limit relation (8). Since h is a homomorphism and q a quadratic form (on $G^\wedge$), we have

$$\begin{aligned}
q(\chi) &= \frac{1}{n^2}q(n\chi) \\
&= \frac{1}{n^2}\psi(n\chi) - \frac{1}{n}ih(\chi) - \frac{c}{n^2} - \frac{1}{n^2}\psi_\mu(n\chi)
\end{aligned}$$

for all $\chi \in G^\wedge$ and $n \geqslant 1$. Hence it suffices to prove that

$$\lim_{n\to\infty} \frac{1}{n^2}\psi_\mu(n\chi) = 0$$

or by the dominated convergence theorem that

$$\lim_{n\to\infty} \frac{1}{n^2}\big(1 - n\chi(x) + \mathrm{i}g(x, n\chi)\big) = 0$$

as well as

$$\frac{1}{n^2}\big|1 - n\chi(x) + \mathrm{i}g(x, n\chi)\big| \leqslant C|1 - \operatorname{Re}\chi(x)|$$

holds for all $x \in G$, $n \geqslant 1$ and some constant $C > 0$. While the above limiting relationship is evident, the inequality requires a look back to the estimate (5) in the proof of Theorem 5.3.5 and an argument in support of the inequality

$$\frac{1}{n^2}(1 - \operatorname{Re} n\chi(x)) \leqslant C(1 - \operatorname{Re}\chi(x))$$

valid for all $x \in G$, $n \geqslant 1$. The latter can be established as follows: For fixed $x \in G$ and $\chi \in G^\wedge$ there exists $\vartheta \in [-\pi, \pi]$ such that $\chi(x) = \mathrm{e}^{\mathrm{i}\vartheta}$. It follows that

$$\begin{aligned}
\frac{1}{n^2}(1 - \operatorname{Re} n\chi(x)) &= \frac{1}{n^2}(1 - \cos(n\vartheta)) \\
&= \frac{2\sin^2 \frac{n\vartheta}{2}}{n^2} \frac{2\sin^2 \frac{\vartheta}{2}}{2\sin^2 \frac{\vartheta}{2}} \frac{(\frac{\vartheta}{2})^2}{(\frac{\vartheta}{2})^2} \\
&= \left(\frac{\sin \frac{n\vartheta}{2}}{\frac{n\vartheta}{2}}\right)^2 \left(\frac{\frac{\vartheta}{2}}{\sin \frac{\vartheta}{2}}\right)^2 (1 - \cos\vartheta) \\
&\leqslant C(1 - \operatorname{Re}\chi(x)),
\end{aligned}$$

since the functions $y \mapsto \frac{\sin y}{y}$ is bounded away from 0 on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Now, by Property 5.4.4.2 the quadruple (c, h, q, μ) is uniquely determined by ψ and deserves to be called canonical. $\hfill\square$

Remark 5.4.8 In the special cases of bounded or symmetric Lévy measures μ of ψ the canonical representation given in Theorem 5.4.7 does not require the use of the Lévy function g which had been introduced in order to achieve the absolute convergence of the integral defining ψ_μ.

Discussion 5.4.9 of the **classical Lévy–Khinchin formula** (See Remark 3.4.21).

In the case $G = G^\wedge = \mathbb{R}^d$ $(d \geqslant 1)$ various Lévy functions (apart from the one introduced in Example 5.3.2.1) can be given. Unfortunately the classical function g_0 defined by

$$g_0(x, y) := \frac{\langle x, y\rangle}{1 + \|x\|^2}$$

for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ is not a Lévy function in the sense of Definition 5.3.1. A slight modification of the above approach, however, makes it possible to obtain the classical analogue of Theorem 5.4.7.

In fact, for a measure $\mu \in M_+((\mathbb{R}^d)^\times)$ the condition

$$\int_{(\mathbb{R}^d)^\times} (1 - \operatorname{Re} e^{-i\langle x, y\rangle})\, \mu(\mathrm{d}x) < \infty$$

valid for all $y \in \mathbb{R}^d$ is equivalent to the condition

$$\int_{U_1 \setminus \{0\}} \|x\|^2\, \mu(\mathrm{d}x) + \int_{\complement U_1} \mu(\mathrm{d}x) < \infty\,,$$

where U_1 denotes the unit ball with center 0 and radius 1. Moreover, it can be shown that for any compact subset K of $\mathbb{R}^d$ there exist a neighbourhood $V \in \mathfrak{V}_{\mathbb{R}^d}(0)$ and a constant $c > 0$ such that

$$\left|1 - e^{i\langle x, y\rangle} + i g_0(x, y)\right| \leqslant c\|x\|^2$$

for all $x \in V$, $y \in K$. This inequality together with the above finiteness condition is the proper replacement for the canonical estimate (5) in the proof of Theorem 5.3.5.

With these tools at hand the previous arguments yield the equivalence of the following statements valid for any complex-valued function ψ on $\mathbb{R}^d$

(i′) $\psi \in \mathrm{CND}(\mathbb{R}^d)$.

(ii′) There is a canonical quadruple (c, h, q, μ') as in (ii) of Theorem 5.4.7 with a bounded Lévy measure $\mu' \in M_+((\mathbb{R}^d)^\times)$ such that

$$\psi(y) = c + i h(y) + q(y)$$
$$+ \int_{(\mathbb{R}^d)^\times} \left(1 - e^{i\langle x, y\rangle} + \frac{i\langle x, y\rangle}{1 + \|x\|}\right) \frac{1 + \|x\|^2}{\|x\|}\, \mu'(\mathrm{d}x)$$

for all $y \in \mathbb{R}^d$.

In order to relate Theorem 5.4.7 for locally compact Abelian groups to the Lévy–Khinchin decomposition theorem 3.4.20 for separable Banach spaces we proceed as follows. As a direct consequence of Theorem 5.4.7 we obtain

Theorem 5.4.10 (Canonical representation of convolution semigroups) *Let g be a Lévy function for the underlying locally compact Abelian group G. Then any convolution semigroup $(\mu_t)_{t \geqslant 0} \in \mathrm{CS}(G)$ has a Fourier representation*

$$\hat{\mu}_t = \exp(-t\psi)$$

valid for all $t > 0$, where the negative definite function $\psi \in \mathrm{CND}(G^\wedge)$ associated with $(\mu_t)_{t \geqslant 0}$ admits a canonical quadruple (c, h, q, μ) of the form (7) in Theorem 5.4.7.

The **proof** of this statement follows from the Schoenberg correspondence theorem 5.2.3 (together with Theorem 5.4.7).

As in Section 2.3 for separable Banach spaces we may also for locally compact Abelian groups G introduce the notions of infinite divisibility and embeddability of probability measures on G.

Definition 5.4.11 A measure $\mu \in M^1(G)$ is called **infinitely divisible** if for every $n \in \mathbb{N}$ there exists an n-th **root** of μ, i.e. a measure $\mu_n \in M^1(G)$ such that

$$\mu_n^n = \mu \,.$$

$\mu \in M^1(G)$ is said to be **embeddable** if there exists a convolution semigroup $(\mu_t)_{t \geqslant 0}$ in $M^1(G)$ such that $\mu_1 = \mu$.

The sets of infinitely divisible and embeddable probability measures on G will be denoted by $\mathrm{I}(G)$ and $\mathrm{EM}(G)$ respectively.

Clearly,

$$\mathrm{EM}(G) \subset \mathrm{I}(G) \,,$$

but the inverse inclusion fails to be true in general. In the special case $G := \mathbb{R}^d$, however, the sets $\mathrm{I}(G)$ and $\mathrm{EM}(G)$ coincide (see the embedding theorem 2.3.9 for arbitrary separable Banach spaces).

In what follows we shall establish the embedding property beyond this special case.

As a preparation the subsequent topological result will be proved.

Lemma 5.4.12 (Existence and uniqueness of the logarithm)

(i) *Let S be a locally arcwise connected topological space, and let $(f_n)_{n \geqslant 1}$ be a sequence of continuous mappings $f_n : S \to \mathbb{C}^\times$ satisfying*

$$f_{n+1}^2 = f_n$$

for all $n \geqslant 1$. Then there exists a continuous complex-valued function h on S with

$$f_1 = \exp h \,.$$

(ii) *Let S be a connected topological space, $a \in S$ and $f : S \to \mathbb{C}^\times$ a
    continuous mapping with $f(a) = 1$.   Then there exists at most one
    continuous complex-valued function h on S with $h(a) = 0$ satisfying*

$$f = \exp h \,.$$

Proof.   (i).   Since every connected component of S is open, we may
assume without loss of generality that S is connected. We fix $a \in S$ and
assume that $f_n(a) = 1$ for all $n \geqslant 1$. Let

$$H := \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\} \,,$$

and let U_n denote the connected component of

$$\{s \in S : f_n(s) \in H\}$$

which contains a. Then $(U_n)_{n \geqslant 1}$ is an increasing sequence of connected
open subsets of S. For every continuous arc $\gamma : [0,1] \to S$ with $\gamma(0) = a$
and sufficiently large n we have $\gamma([0,1]) \subset U_n$ or $S = \cup_{n \geqslant 1} U_n$.

Indeed, there exists a continuous complex-valued function φ on $[0,1]$
with $f_1 \circ \gamma = \exp \varphi$. But then, for large n and $\psi := 2^{1-n} \varphi$ the function
$f_n \circ \gamma = \exp \psi$ takes values in H.

Now, let $\log$ denote the principal branch of the logarithm on H and
define h_n on U_n by

$$h_n := 2^{n-1} \log f_n$$

for each $n \geqslant 1$. The function h_n on U_n is continuous and coincides
with h_{n+1} on U_n. Thus there exists a function h on S satisfying
$\mathrm{Res}_{U_n} h = h_n$ for all $n \geqslant 1$. h is the desired logarithm.

(ii).   Let h and h' be two continuous complex-valued functions on
S with $h(a) = h'(a)$ and

$$f = \exp h = \exp h' \,.$$

Then the continuous function $h' - h$ takes only values in $2\pi \mathbb{Z}$ and is
therefore constant. Consequently $h' = h$ since $h'(a) - h(a) = 0$. $\square$

Summary 5.4.13 Let S be a connected and locally arcwise connected
topological space, and let $a \in S$.   Moreover, let $f : S \to \mathbb{C}^\times$ be a
continuous mapping with $f(a) = 1$. Suppose that for every $n \geqslant 1$ there
exists a continuous mapping $g_n : S \to \mathbb{C}^\times$ such that

$$g_n^n = f \,.$$

Then there exists a unique continuous complex-valued function h on S
with $h(a) = 0$ satisfying

$$f = \exp h \,.$$

In particular $f \neq 0$ on S.

Theorem 5.4.14 (Embedding of an infinitely divisible probability measure)

Let G be a second countable locally compact Abelian group with arcwise connected dual $G^\wedge$. Then

$$\mathrm{I}(G) \subset \mathrm{EM}(G).$$

Moreover, for each $\mu \in \mathrm{I}(G)$ there exists a unique convolution semigroup $(\mu_t)_{t \geqslant 0}$ in $M^1(G)$ such that $\mu_1 = \mu$.

Proof. By hypothesis G is second countable, hence $G^\wedge$ is also second countable (see [63], Chapter 4, 2.3).

Since $G^\wedge$ is assumed to be arcwise connected, $G^\wedge$ is connected and locally arcwise connected (see [16]).

Now, let μ be a measure in $\mathrm{I}(G)$. Summary 5.4.13 applies to the space $S := G^\wedge$ and to the function $f = \widehat{\mu}$ so that $\widehat{\mu}(\chi) \neq 0$ for all $\chi \in G^\wedge$ and that there exists a continuous branch of $\log \widehat{\mu}$ with $\log \widehat{\mu}(0) = 0$.

Consequently we have for the n-th roots $\mu_{\frac{1}{n}}$ of μ that

$$\widehat{\mu_{\frac{1}{n}}}^{\,n} = \widehat{\mu} = \exp(\log \widehat{\mu})$$

and hence that

$$\widehat{\mu_{\frac{1}{n}}} = \exp\left(\frac{1}{n} \log \widehat{\mu}\right)$$

on $G^\wedge$. We see that the n-th roots $\mu_{\frac{1}{n}}$ of μ are uniquely determined. For each $r := \frac{p}{q} \in \mathbb{Q}_+^\times$ let

$$\mu_{\frac{p}{q}} = \left(\mu_{\frac{1}{q}}\right)^p.$$

Then

$$\widehat{\mu_{\frac{p}{q}}} = \exp\left(\frac{1}{q} \log \widehat{\mu}\right)^p = \exp\left(\frac{p}{q} \log \widehat{\mu}\right)$$

on $G^\wedge$. For $\frac{p}{q} = \frac{p'}{q'} \in \mathbb{Q}_+^\times$ we get

$$\widehat{\mu_{\frac{p'}{q'}}} = \exp\left(\frac{p'}{q'} \log \widehat{\mu}\right) = \exp\left(\frac{p}{q} \log \widehat{\mu}\right) = \widehat{\mu_{\frac{p}{q}}}$$

on $G^\wedge$. The notation $\mu_r := \mu_{\frac{p}{q}}$ is therefore justified, and for all $r, r' \in \mathbb{Q}_+^\times$ the equalities

$$(\mu_r * \mu_{r'})^\wedge = \widehat{\mu_r} \, \widehat{\mu_{r'}}$$
$$= \exp\left((r + r') \log \widehat{\mu}\right)$$
$$= \widehat{\mu_{r+r'}}$$

hold on $G^\wedge$. With $\mu_0 := \varepsilon_0$ we obtain a (rational) convolution semigroup $(\mu_r)_{r \in \mathbb{Q}}$ in $M^1(G)$.

Next we show that the mapping

$$r \mapsto \mu_r$$

from $\mathbb{Q}_+$ into $M^1(G)$ is τ_{w}-continuous.

Let $(r_n)_{n \geqslant 1}$ be a sequence in $\mathbb{Q}_+$ with $r_n \to r \in \mathbb{Q}$. Then

$$\tau_{\mathrm{co}}-\lim_{n\to\infty} \widehat{\mu_{r_n}} = \tau_{\mathrm{co}}-\lim_{n\to\infty} \exp(r_n \log \widehat{\mu})$$
$$= \exp(r \log \widehat{\mu})$$
$$= \widehat{\mu_r}$$

on $G^\wedge$. By Theorem 4.3.7 this implies that

$$\tau_{\mathrm{w}}-\lim_{n\to\infty} \mu_{r_n} = \mu\,.$$

Finally, we prove that $(\mu_r)_{r \in \mathbb{Q}_+}$ extends to a unique convolution semigroup $(\mu_t)_{t \in \mathbb{R}_+}$ in $M^1(G)$ with $\mu_1 = \mu$.

Let $r_n \to t \in \mathbb{R}_+$ with $r_n \in \mathbb{Q}_+$ for all $n \geqslant 1$. Then

$$\widehat{\mu_{r_n}} \to \exp(t \log \widehat{\mu})$$

on $G^\wedge$. But the limiting function

$$\chi \mapsto \exp(t \log \widehat{\mu}(\chi))$$

on $G^\wedge$ takes on the value 1 at the unit character 0 of $G^\wedge$, and belongs to $\mathrm{CPD}(G^\wedge)$ for every $t \in \mathbb{R}_+$ by Property 4.3.3.4. Thus, by Bochner's theorem (Remark 4.3.6) there exists a unique measure $\mu_t \in M^1(G)$ such that

$$\widehat{\mu_t} = \exp(t \log \widehat{\mu})$$

on $G^\wedge$. Clearly,

$$(\mu_t * \mu_{t'})^\wedge = \widehat{\mu_t}\,\widehat{\mu_{t'}}$$
$$= \exp\big((t + t') \log \widehat{\mu}\big)$$
$$= \widehat{\mu_{t+t'}}$$

on $G^\wedge$ for all $t, t' \in \mathbb{R}_+$, hence $(\mu_t)_{t \in \mathbb{R}_+}$ is a convolution semigroup in $M^1(G)$.

The uniqueness of this embedding convolution semigroup follows from the τ_{w}-continuity of the mapping $r \mapsto \mu_r$ from $\mathbb{Q}_+$ into $M^1(G)$. $\qquad\square$

Combining Theorem 5.4.14 with Theorems 5.4.10 and 5.4.7 we obtain the following

Theorem 5.4.15 (Canonical representation of an infinitely divisible probability measure)
Let G be a second countable locally compact Abelian group with locally arcwise connected dual $G^\wedge$.

Any measure $\mu \in I(G)$ with embedding convolution semigroup $(\mu_t)_{t \geqslant 0} \in CS(G)$ has a Fourier representation

$$\hat{\mu} = \exp(-\psi),$$

where $\psi \in CND(G^\wedge)$ associated with $(\mu_t)_{t \in \mathbb{R}_+}$ admits a canonical quadruple of the form (7) in Theorem 5.4.7.

Remark 5.4.16 It is a consequence of a theorem of Dixmier in [16] that the groups for which the embedding and canonical representation results hold, are **aperiodic** in the sense that they admit no proper compact subgroup and hence are closed subgroups of locally convex vector spaces of the form

$$\mathbb{R}^n \times \mathbb{Z}^I$$

(embeddable into $\mathbb{R}^{n+|I|}$), where $n \geqslant 0$ and I is a countable set.

Chapter 6

Probabilistic Properties of Convolution Semigroups

6.1 Transient convolution semigroups

The probabilistic notions of transience and recurrence for stationary independent increment processes can be discussed in terms of convolution semigroups. This analytic treatment yields further interesting results of potential-theoretic nature.

Let G denote a locally compact Abelian group. For a convolution semigroup $(\mu_t)_{t\geqslant 0}$ in $\mathrm{CS}(G)$ with associated resolvent $(\varrho_\lambda)_{\lambda>0}$ and any function $f \in C_+^{\mathrm{c}}(G)$ we observe that the mapping

$$\lambda \mapsto \int f \,\mathrm{d}\varrho_\lambda = \int_0^\infty \mathrm{e}^{-\lambda t} \mu_t(f) \,\mathrm{d}t$$

for $\mathbb{R}_+^\times$ into $\mathbb{R}$ is decreasing. Hence the monotone convergence theorem implies that

$$\lim_{\lambda\to 0} \int f \,\mathrm{d}\varrho_\lambda = \int_0^\infty \mu_t(f) \,\mathrm{d}t \leqslant \infty\,.$$

Definition 6.1.1 A convolution semigroup $(\mu_t)_{t\geqslant 0}$ in $\mathrm{CS}(G)$ is called **transient** if

$$\lim_{\lambda\to 0} \int f \,\mathrm{d}\varrho_\lambda = \int_0^\infty \mu_t(f) \,\mathrm{d}t < \infty$$

for all $f \in C_+^{\mathrm{c}}(G)$, and in this case

$$\kappa := \tau_{\mathrm{v}}\text{-}\lim_{\lambda\to 0} \varrho_\lambda \in M_+(G)$$

is said to be the **potential measure** of $(\mu_t)_{t\geqslant 0}$.

$(\mu_t)_{t\geqslant 0}$ is called **recurrent** if $(\mu_t)_{t\geqslant 0}$ is not transient.

Discussion 6.1.2

6.1.2.1 Theorem 5.2.16 and its proof imply that the potential measure κ of a transient convolution semigroup $(\mu_t)_{t \geqslant 0} \in \mathrm{CS}(G)$ with associated resolvent $(\varrho_\lambda)_{\lambda > 0}$ has the property that

$$\mathrm{supp}(\kappa) = \mathrm{supp}(\varrho_\lambda) = \left(\bigcup_{t > 0} \mathrm{supp}(\mu_t) \right)^{-}$$

(for all $\lambda > 0$) and that $\mathrm{supp}(\kappa)$ is a semigroup in G.

6.1.2.2 Let $(\mu_t)_{t \geqslant 0} \in \mathrm{CS}(G)$ and $\psi \in \mathrm{CND}(G^\wedge)$ with

$$(\mu_t)_{t > 0} \leftrightarrow \psi \,.$$

If $\psi(0) > 0$ then $(\mu_t)_{t \geqslant 0}$ is transient, $\kappa \in M_+^{\mathrm{b}}(G)$ with

$$\|\kappa\| = \frac{1}{\psi(0)} \,,$$

and

$$\hat{\kappa} = \frac{1}{\psi} \,.$$

On the other hand we shall see later, that there exist transient as well as recurrent convolution semigroups $(\mu_t)_{t \geqslant 0}$ in $M^1(G)$ with

$$(\mu_t)_{t \geqslant 0} \leftrightarrow \psi$$

such that $\psi(0) = 0$. In fact, for transient convolution semigroups $(\mu_t)_{t \geqslant 0}$ in $M^1(G)$ the potential measure κ is always unbounded.

6.1.2.3 If G is compact, a convolution semigroup $(\mu_t)_{t \geqslant 0} \in \mathrm{CS}(G)$ with

$$(\mu_t)_{t \geqslant 0} \leftrightarrow \psi \,,$$

where $\psi \in \mathrm{CND}(G^\wedge)$, is transient if and only if $\psi(0) > 0$. This equivalence holds true for an arbitrary locally compact Abelian group G provided there exists a compact subset K of G such that

$$\mathrm{supp}(\mu_t) \subset K$$

for all $t \geqslant 0$.

We add an important

Example 6.1.3 Let $(\mu_t)_{t \geqslant 0}$ be the Poisson semigroup determined by a measure $\mu \in M_+^{(1)}(G)$ introduced in Example 5.2.7.2. We recall that

$$\mu_t = \mathrm{e}^{-t} \exp(t\mu)$$

for all $t \geqslant 0$.

(μ_t) is transient if and only if the series

$$\sum_{n \geqslant 0} \mu^n \text{ is } \tau_v\text{-convergent.}$$

In this case $\sum_{n \geqslant 0} \mu^n \in M_+(G)$, and as such it coincides with the potential measure κ of $(\mu_t)_{t \geqslant 0}$.

In fact, for the resolvent $(\varrho_\lambda)_{\lambda \geqslant 0}$ with

$$(\varrho_\lambda)_{\lambda > 0} \leftrightarrow (\mu_t)_{t \geqslant 0}$$

we obtain the following equalities for Radon measures

$$\varrho_\lambda = \int_0^\infty e^{-\lambda t} e^{-t} \exp(t\mu) \, dt$$

$$= \int_0^\infty e^{-(\lambda+1)t} \left(\sum_{n \geqslant 0} \frac{t^n}{n!} \mu^n \right) dt$$

$$= \sum_{n \geqslant 0} \mu^n \frac{1}{n!} \int_0^\infty t^n e^{-(\lambda+1)t} \, dt$$

$$= \sum_{n \geqslant 0} \left(\frac{1}{\lambda+1} \right)^{n+1} \mu^n,$$

hence for $\lambda \to 0$ the assertion.

We note that measures $\kappa \in M_+(G)$ of the form

$$\kappa := \sum_{n \geqslant 0} \mu^n$$

with $\mu \in M_+^{(1)}(G)$ are called ***elementary kernels*** determined by μ.

Properties 6.1.4 of potential measures of transient convolution semigroups $(\mu_t)_{t \geqslant 0} \in \mathrm{CS}(G)$ with associated $\psi \in \mathrm{CND}(G^\wedge)$ and resolvent $(\varrho_\lambda)_{\lambda > 0}$.

6.1.4.1 *For every* $\lambda > 0$ *we have that*

$$\frac{1}{2}(\varrho_\lambda + \varrho_\lambda^\sim) \in M_{\mathrm{p}}(G),$$

hence that

$$\frac{1}{2}(\kappa + \kappa^\sim) \in M_{\mathrm{p}},$$

and $\kappa \in M^{\mathrm{sb}}(G)$.

6.1.4.2 $\mathrm{Inv}(\kappa) = \{0\}$

6.1.4.3 *For each* $\lambda > 0$ *the measure*

$$\lambda\kappa + \varepsilon_0$$

is the elementary kernel determined by the measure $\lambda\varrho_\lambda$.

Both Properties 6.1.4.1 and 6.1.4.3 are proved by taking Fourier transforms. For Property 6.1.4.1 we just note that $\frac{1}{2}(\varrho_\lambda + \varrho_\lambda^{\sim}) \in M_{\mathrm{p}}(G)$, since

$$\mathcal{F}_G\left[\frac{1}{2}(\varrho_\lambda + \varrho_\lambda{}^{\sim})\right] = \mathrm{Re}\,\frac{1}{\lambda + \psi}$$
$$= \frac{\lambda + \mathrm{Re}\,\psi}{|\lambda + \psi|^2} \geqslant 0$$

(see Theorem 4.4.14). The τ_{v}-closedness of $M_{\mathrm{p}}(G)$ in $M(G)$ implies that $\frac{1}{2}(\kappa + \kappa^{\sim}) \in M_{\mathrm{p}}(G)$, and the assertion follows from Proposition 4.4.9 (ii).

In the proof of Property 6.1.4.3 one verifies the equality

$$\varrho_{\lambda'} = \sum_{n \geqslant 0}(\lambda - \lambda')^n {\varrho_\lambda}^{n+1}$$

for $\lambda' \in]0, \lambda[$ by applying the Fourier transform and then takes the limit for $\lambda' \downarrow 0$.

Terminology 6.1.5 Let $\psi \in \mathrm{CND}(G^\wedge)$.

(a) The function $\frac{1}{\psi}$ is said to be **integrable over some** measurable **subset** W of $G^\wedge$ provided

$$\psi \neq 0 \ [\omega_{G^\wedge}] \ \text{ on } \ W$$

and $\frac{1}{\psi}$ (defined $[\omega_{G^\wedge}]$ on W) is $\omega_{G^\wedge}$-integrable over W.
(b) The real function $\mathrm{Re}\,\frac{1}{\psi}$ is said to be **locally integrable** (on $G^\wedge$) if there exists a neighbourhood $U \in \mathfrak{V}_{G^\wedge}(0)$ such that

$$\psi \neq 0 \ [\omega_{G^\wedge}] \ \text{ on } \ U$$

and $\mathrm{Re}\,\frac{1}{\psi}$ is $\mathrm{Res}_U\,\omega_{G^\wedge}$-integrable.

We note that $\mathrm{Re}\,\frac{1}{\psi}$ is locally integrable if and only if it is integrable over every open, relatively compact subset of $G^\wedge$.

With this slightly sophisticated terminology in mind we are able to establish a generalization of Property 5.1.7.1.

Proposition 6.1.6 *Let* $\psi \in \mathrm{CND}(G^\wedge)$ *and assume that* $\frac{1}{\psi}$ *is integrable over some open, relatively compact neighbourhood* $\in \mathfrak{V}_{G^\wedge}(0)$. *Then*

(i) $\frac{1}{\psi}$ is locally integrable

and

(ii) $\frac{1}{\psi} \cdot \omega_{G^\wedge} \in M_{\mathrm{p}}(G)$.

Proof. From the discussion 6.1.2.2 we know that it suffices to consider the case that $\psi(0) = 0$. Let K be a compact subset of $G^\wedge$. If $\psi(\chi) \neq 0$ for all $\chi \in K$, we have that

$$\int_K \frac{1}{|\psi(\chi)|} \omega_{G^\wedge}(\mathrm{d}\chi) < \infty \,.$$

Now let $\psi(\chi) = 0$ for some $\chi \in K$. Then the set

$$K_1 := \{\chi \in K : \psi(\chi) = 0\}$$

is compact, hence

$$K_1 \subset \bigcup_{i=1}^n (\chi_i + V) \,,$$

where $\chi_1, \ldots, \chi_n \in K_1$ and V is an open, relatively compact neighbourhood $\in \mathfrak{V}_{G^\wedge}(0)$ such that $\frac{1}{\psi}$ is integrable over V. The set

$$F := K \setminus \bigcup_{i=1}^n (\chi_i + V)$$

being compact with $\psi(\chi) \neq 0$ for all $\chi \in F$ we see that $\psi \neq 0 \; [\omega_{K^\wedge}]$ on K, and that

$$\int_K \frac{1}{|\psi(\chi)|} \omega_{K^\wedge}(\mathrm{d}\chi) \leqslant \int_F \frac{1}{|\psi(\chi)|} \omega_{G^\wedge}(\mathrm{d}\chi) + \int_{\bigcup_{i=1}^n (\chi_i+V)} \frac{1}{|\psi(\chi)|} \omega_{G^\wedge}(\mathrm{d}\chi)$$

$$\leqslant \int_F \frac{1}{|\psi(\chi)|} \omega_{G^\wedge}(\mathrm{d}\chi) + n \int_V \frac{1}{|\psi(\chi)|} \omega_{G^\wedge}(\mathrm{d}\chi) < \infty \,.$$

Here we employed the fact that $K_1 \subset \mathrm{Inv}(\psi)$ which is part of Theorem 5.2.17.

We turn to the proof of statement (ii) of the theorem. For a given $f \in C^c(G^\wedge)$ and $n \geqslant 1$ we have

$$\int f * f^\sim \frac{1}{\psi + \frac{1}{n}} \, \mathrm{d}\omega_{G^\wedge} \geqslant 0 \,,$$

since $\frac{1}{\psi + \frac{1}{n}} \in \mathrm{CPD}(G^\wedge)$ by Property 5.1.7.1. The function $f * f^\sim \frac{1}{\psi}$ is defined $[\omega_{G^\wedge}]$, and

$$\lim_{n \to \infty} f * f^\sim \frac{1}{\psi + \frac{1}{n}} = f * f^\sim \frac{1}{\psi} \; [\omega_{G^\wedge}] \; (\text{on } G^\wedge) \,.$$

But $\operatorname{Re}\psi \geqslant 0$ implies that

$$|\psi| \leqslant \left|\psi + \frac{1}{n}\right|$$

for all $n \geqslant 1$, hence Lebesgue's dominated convergence theorem yields the assertion (ii). $\qquad\qquad\square$

Remark 6.1.7 Proposition 6.1.6 remains valid if one replaces $\frac{1}{\psi}$ by $\operatorname{Re}\frac{1}{\psi}$ with the modification that (ii) reads as $\operatorname{Re}\frac{1}{\psi} \cdot \omega_{G^\wedge} \in M_+(G^\wedge)$.

We are now aiming at a characterization of transience of convolution semigroups in terms of integrability of their associated negative definite functions. The following statement takes care of the less sophisticated part of that equivalence.

Theorem 6.1.8 *Let* $(\mu_t)_{t\geqslant 0} \in \operatorname{CS}(G)$ *and* $\psi \in \operatorname{CND}(G^\wedge)$ *with*

$$(\mu_t)_{t\geqslant 0} \leftrightarrow \psi\,.$$

If $(\mu_t)_{t\geqslant 0}$ *is transient, then* $(\mu_t)_{t\geqslant 0}$ *is **locally integrable** in the sense that* $\operatorname{Re}\frac{1}{\psi}$ *is locally integrable.*

Proof. Without loss of generality we assume that $\psi(0) = 0$. We know that the set

$$T := \{\chi \in G^\wedge : \psi(\chi) = 0\}$$

is a closed subgroup of $G^\wedge$ and by Theorem 5.2.17 (ii) that

$$\operatorname{supp}(\mu_t) \subset T^\perp$$

for all $t > 0$. Since $(\mu_t)_{t\geqslant 0}$ is transient by assumption, $T^\perp$ is not compact, hence T is not open. But a closed subgroup of $G^\wedge$ is either open or a local $\omega_{G^\wedge}$-null set. This implies that $\psi \neq 0$ locally $[\omega_{G^\wedge}]$ on $G^\wedge$. Now, let K be a compact subset of $G^\wedge$. As usual we choose a function $f \in C_+^c(G)$ with

$$|\overline{\mathcal{F}}_G f| \geqslant 1 \quad \text{on}\ \ K.$$

But then the monotone convergence theorem together with formula (4) in

the proof of Theorem 4.4.14 yields

$$
\begin{aligned}
\int_K \operatorname{Re} \frac{1}{\psi} \, \mathrm{d}\omega_{G^\wedge}
&= \int_K \lim_{\lambda \to 0} \operatorname{Re} \frac{1}{\lambda + \psi} \, \mathrm{d}\omega_{G^\wedge} \\
&\leqslant \int_{G^\wedge} \liminf_{\lambda \to 0} |\overline{\mathcal{F}}_G f|^2 \operatorname{Re} \frac{1}{\lambda + \psi} \, \mathrm{d}\omega_{G^\wedge} \\
&\leqslant \liminf_{\lambda \to 0} \int_{G^\wedge} |\overline{\mathcal{F}}_G f|^2 \operatorname{Re} \frac{1}{\lambda + \psi} \, \mathrm{d}\omega_{G^\wedge} \\
&= \liminf_{\lambda \to 0} \int_{G^\wedge} |\overline{\mathcal{F}}_G f|^2 \mathcal{F}_G \left[\frac{1}{2}(\varrho_\lambda + \varrho_\lambda{}^\sim) \right] \, \mathrm{d}\omega_{G^\wedge} \\
&= \liminf_{\lambda \to 0} \frac{1}{2} \int_G f * f^\sim \mathrm{d}(\varrho_\lambda + \varrho_\lambda^\sim) \\
&= \frac{1}{2} \int_G f * f^\sim \, \mathrm{d}(\kappa + \kappa^\sim) < \infty \,,
\end{aligned}
$$

where, as in the proof of Property 6.1.4.1, the identity

$$
\mathcal{F}_G \left[\frac{1}{2}(\varrho_\lambda + \varrho_\lambda^\sim) \right] = \operatorname{Re} \frac{1}{\lambda + \psi}
$$

has been applied (for $\lambda > 0$), $(\varrho_\lambda)_{\lambda > 0}$ and κ being as usual associated with $(\mu_t)_{t \geqslant 0}$. $\qquad\square$

Corollary 6.1.9 *Let* $\mu \in M_+^{(1)}(G)$ *be such that*

$$
\sum_{n \geqslant 0} \mu^n \quad \text{is} \quad \tau_v\text{-convergent.}
$$

Then $\operatorname{Re} \frac{1}{1 - \hat{\mu}}$ *is locally integrable (on* $G^\wedge$ *).*

Proof. We look at the convolution semigroup $(\mu_t)_{t \geqslant 0}$ determined by the measure μ and its associated function $\psi = 1 - \hat{\mu} \in \mathrm{CND}(G^\wedge)$. By Example 6.1.3 $(\mu_t)_{t \geqslant 0}$ is transient, hence the theorem implies the assertion. $\qquad\square$

The discussion in the remaining part of this section will be devoted to the implication (inverse to that of Theorem 6.1.8)

(TC) $(\mu_t)_{t \geqslant 0}$ locally integrable $\Rightarrow$ $(\mu_t)_{t \geqslant 0}$ transient.

Remark 6.1.10 Until now the implication (TC) is only available in the special case of a compact group G. See Discussion 6.1.2.3.

In other words, (TC) holds for convolution semigroups $(\mu_t)_{t \geqslant 0}$ in $M_+^b(G)$ which are not in $M^1(G)$.

If, in fact, $\|\mu_t\| < 1$ for some $t > 0$, then $(\mu_t)_{t \geqslant 0}$ is transient, and for the associated $\psi \in \mathrm{CND}(G^{\wedge})$ we have $\mathrm{Re}\,\psi > 0$ by Corollary 5.2.5 (ii).

For the somewhat lengthy proof of the implication (TC) which in its course will contain useful side results on transient convolution semigroups in $M_+^{(1)}(G)$ we shall prepare several steps of reducing the statement to be shown and two basic facts from the potential theory of convolution semigroups in $M_+^{(1)}(G)$.

Properties 6.1.11 (Reducing the proof of (TC))

6.1.11.1 *Let H be a compact subgroup of G and let $(\dot{\mu}_t)_{t \geqslant 0} \in \mathrm{CS}(G/H)$ be the canonical image of $(\mu_t)_{t \geqslant 0} \in \mathrm{CS}(G)$. Then (TC) holds for $(\mu_t)_{t \geqslant 0}$ provided it holds for $(\dot{\mu}_t)_{t \geqslant 0}$.*

In fact, given a recurrent convolution semigroup $(\mu_t)_{t \geqslant 0} \in \mathrm{CS}(G)$, $(\dot{\mu}_t)_{t \geqslant 0}$ is a recurrent convolution semigroup $\in \mathrm{CS}(G/H)$. Since $(G/H)^{\wedge}$ is isomorphic to the open subgroup $H^{\perp}$ of $G^{\wedge}$ by the functorial property 4.2.17.3 and since $\dot{\psi} \in \mathrm{CND}((G/H)^{\wedge})$ with

$$\dot{\psi} \leftrightarrow (\dot{\mu}_t)_{t \geqslant 0}$$

coincides with $\mathrm{Res}_{H^{\perp}}\,\psi$, where

$$\psi \leftrightarrow (\mu_t)_{t \geqslant 0}\,,$$

$\mathrm{Re}\,\frac{1}{\psi}$ is not locally integrable.

6.1.11.2 *Without loss of generality the given $(\mu_t)_{t \geqslant 0} \in \mathrm{CS}(G)$ can be assumed to be **adapted** in the sense that*

$$S := \left[\bigcup_{t > 0} \mathrm{supp}(\mu_t)\right]^{-} = G\,.$$

In fact, if $(\mu_t)_{t \geqslant 0}$ is not adapted, then we replace G by S. Considering the canonical surjection $\pi : G^{\wedge} \to G^{\wedge}/S^{\perp} \cong S^{\wedge}$ (functorial property 4.2.17.4) we obtain that

$$\psi = \dot{\psi} \circ \pi$$

holds for $\psi \in \mathrm{CND}(G^{\wedge})$ with $\psi \leftrightarrow (\mu_t)_{t \geqslant 0}$ and $\dot{\psi} \in \mathrm{CND}(S^{\wedge})$ with

$$\dot{\psi} \leftrightarrow (\dot{\mu}_t)_{t \geqslant 0}\,.$$

But then $\mathrm{Re}\,\frac{1}{\psi}$ is locally integrable on $G^{\wedge}$ if and only if $\mathrm{Re}\,\frac{1}{\dot{\psi}}$ is locally integrable on $S^{\wedge}$.

6.1.11.3 *It is sufficient to prove* (TC) *for adapted convolution semigroups in* $\mathrm{CS}(G)$, *where* G *is of the form*

$$G = \mathbb{R}^d \times \mathbb{Z}^e$$

for $d, e \geqslant 0$.

This follows from the structure theorem 4.2.20 which implies that

$$G/H \cong \mathbb{R}^d \times \mathbb{Z}^e$$

for some compact subgroup H of G and integers $d, e \geqslant 0$. In fact, since every closed subgroup of a group of the form $\mathbb{R}^d \times \mathbb{Z}^e$ is of the same type (being a closed subgroup of $\mathbb{R}^{d+e}$), the previous properties yield the assertion.

Finally

6.1.11.4 *it is sufficient to establish* (TC) *for (adapted) Poisson semigroups on* $G = \mathbb{R}^d \times \mathbb{Z}^e$ *for* $d, e \geqslant 0$.

In order to see this one employs the resolvent $(\varrho_\lambda)_{\lambda > 0}$ associated with $(\mu_t)_{t \geqslant 0}$ and considers the Poisson semigroup $(\mathrm{e}(t\varrho_1))_{t \geqslant 0}$ determined by $\varrho_1 \in M_+^{(1)}(G)$. It is easily shown that (TC) for $(\mu_t)_{t \geqslant 0}$ is a consequence of the implication

$$(\mathrm{e}(t\varrho_1))_{t \geqslant 0} \ \text{ locally integrable } \Rightarrow \sum_{n \geqslant 0} \varrho_1^n \ \text{ is } \ \tau_v\text{-convergent.}$$

Moreover,

$$[\mathrm{supp}(\varrho_1)]^- = G$$

by theorem 5.2.16, once $(\mu_t)_{t \geqslant 0}$ is adapted.

Theorem 6.1.12 (The Choquet–Deny **convolution equation**)
Let G *be a locally compact Abelian group, and let* $\sigma \in M^1(G)$ *be a fixed measure with* **support subgroup**

$$G(\sigma) := [\mathrm{supp}(\sigma)]^- .$$

Then, for every measure $\mu \in M_+(G)$ *the following statements are equivalent:*

(i) $\mu \in M^{\mathrm{sb}}(G)$ *and*

$$\mu * \sigma = \mu .$$

(ii) $\mathrm{Inv}(\mu) \supset G(\sigma)$.

Proof.   It suffices to verify the implication (i) $\Rightarrow$ (ii). Let $\mu \in M_+(G)$ be a measure satisfying (i) of the theorem. For $\varphi \in C^c(G)$ we introduce the function $f := f_\varphi$ on G defined by

$$f(x) := \int \varphi(x + y)\,\mu(\mathrm{d}y)$$

whenever $x \in G$. Clearly, f is uniformly continuous (with respect to the uniform structure of G), and it is bounded since $\mu \in M^{\mathrm{sb}}(G)$. Moreover, f satisfies the integral equations

$$f(x) = \int f(x + z)\,\sigma(\mathrm{d}z)$$

for all $x \in G$. Then for $a \in \mathrm{supp}(\sigma)$ the functions $T_{-a}f$ and $g := T_{-a}f - f$ are uniformly continuous, bounded and satisfy the above integral equation.

Now let $c \in \mathbb{R}_+$ be such that $|g| \leqslant c$ and let

$$\gamma := \frac{1}{2}\sup_{x \in G} g(x)\,.$$

There is a sequence $(x_n)_{n \geqslant 1}$ in G satisfying

$$\lim_{n \to \infty} g(x_n) = 2\gamma\,.$$

Then the set $\{g_n : n \in \mathbb{N}\}$ of functions $g_n := T_{-x_n}g$ $(n \geqslant 1)$ is equicontinuous, since g is uniformly continuous. Hence, by the Arzelà–Ascoli theorem there exists a subnet $(g_{n_\alpha})_{\alpha \in A}$ of $(g_n)_{n \geqslant 1}$ which converges with respect to the topology τ_{co} towards a function $h \in C(G)$ with $|h| \leqslant c$. Furthermore h satisfies the above integral equation. But then

$$\begin{aligned}
2\gamma &= \lim_{\alpha \in A} g(x_{n_\alpha}) \\
&= \lim_{\alpha \in A} g_{n_\alpha}(e) \\
&= h(e) \\
&= \int h\,\mathrm{d}\sigma\,.
\end{aligned}$$

From $h \leqslant 2\gamma$ we conclude that $h(x) = 2\gamma$ for all $x \in \mathrm{supp}(\sigma)$. Given $x \in \mathrm{supp}(\sigma)$ we further obtain

$$2\gamma = h(x) = \int h(x + z)\,\sigma(\mathrm{d}z)\,,$$

hence

$$h(x + z) = 2\gamma$$

for all $z \in \text{supp}(\sigma)$. An iteration procedure yields

$$h(x) = 2\gamma$$

for all x belonging to the closed semigroup generated by $\text{supp}(\sigma)$ and therefore

$$h(ka) = 2\gamma$$

for all $a \in \text{supp}(\sigma)$, $k \geqslant 1$. Now, for every $l \geqslant 1$ there exists an $\alpha \in A$ with

$$g_{n_\alpha}(ka) > \gamma$$

whenever $k = 1, \dots, l$, provided $\gamma > 0$. But then by looking at the equalities

$$\begin{aligned} g_{n_\alpha}(ka) &= f(x_{n_\alpha} + ka + a) - f(x_{n_\alpha} + ka) \\ &= f(x_{n_\alpha} + (k+1)a) - f(x_{n_\alpha} + ka), \end{aligned}$$

and after summing for $k = 1, \dots, l$ we obtain

$$f(x_{n_\alpha} + (l+1)a) - f(x_{n_\alpha} + a) > l\gamma.$$

Since f is bounded, this yields a contradiction. Therefore $\gamma \leqslant 0$, hence $g \leqslant 0$. Replacing g by $-g$ one gets $-g \leqslant 0$, thus altogether $g = 0$ or

$$f = T_{-a}f$$

for all $a \in \text{supp}(\sigma)$. From this follows

$$\int \varphi(y + a)\, \mu(\mathrm{d}y) = \int \varphi(y)\, \mu(\mathrm{d}y),$$

hence

$$\mu * \varepsilon_a = \mu$$

for all $a \in \text{supp}(\sigma)$ and finally for all $a \in G(\sigma)$. $\square$

Corollary 6.1.13 (Characterizing **idempotent measures**)
For any $\mu \in M_+^{\mathrm{b}}(G) \setminus \{0\}$ the following statement are equivalent:

(i) $\mu * \mu = \mu$.
(ii) $\mu = \omega_H$ *for some compact subgroup H of G.*

Proof. Again it suffices to show the implication (i) $\Rightarrow$ (ii). Without loss of generality we may assume that $\mu \in M^1(G)$, since for $\mu \in M^b_+(G) \setminus \{0\}$ one has

$$\|\mu * \mu\| = \|\mu\|\|\mu\| = \|\mu\|,$$

hence $\|\mu\| = 1$.

From the boundedness of μ follows that $\mathrm{Inv}(\mu)$ is a compact subset of G (see the discussion following Definition 4.4.17). Now, Theorem 6.1.12 says that

$$G(\mu) \subset \mathrm{Inv}(\mu),$$

hence that $H := G(\mu)$ is a compact subgroup of G and μ is H-invariant on H. Thus $\mu = \omega_H$ (the normed Haar measure of H). $\qquad\square$

The following result from potential theory of convolution semigroups goes back to Deny and has been reproved in the book [8] of Berg and Forst.

Theorem 6.1.14 (Existence of an equilibrium measure)
*Let G be a locally compact Abelian group, $(\mu_t)_{t \geqslant 0}$ a convolution semi-group $\in \mathrm{CS}(G)$ with potential measure $\kappa \in M_+(G)$, and let W be an open, relatively compact subset of G. Then there exists an **equilibrium measure** $\gamma \in M^b_+(G)$ for $(\mu_t)_{t \geqslant 0}$ which by definition has the following properties:*

(a) $\mathrm{supp}(\gamma) \subset \overline{W}$
(b) $\kappa * \gamma \leqslant \omega_G$
(c) $\kappa * \gamma = \omega_G$ *on* W.

Corollary 6.1.15 *For every $\sigma \in M^{(1)}_+(G)$ and $f \in C^c_+(G)$ with $\mathrm{supp}(f^\sim) \subset W$ we have that*

$$\kappa * \gamma * (\varepsilon_0 - \sigma) * f(0) \geqslant 0.$$

The proof follows immediately from the inequalities

$$\big(\kappa * \gamma * (\varepsilon_0 - \sigma) * f\big)(0) = (\kappa * \gamma * f)(0) - (\kappa * \gamma * \sigma * f)(0)$$
$$= \int f^\sim \,\mathrm{d}\omega_G - \int f^\sim \,\mathrm{d}(\kappa * \gamma * \sigma) \geqslant 0,$$

since

$$\kappa * \gamma * \sigma \leqslant \omega_G * \sigma \leqslant \omega_G.$$

6.2   The transience criterion

This section will be devoted entirely to the announced measure-theoretic proof of the famous transience criterion for convolution semigroups which for second countable locally compact Abelian groups and with probabilistic methods has been established by Port and Stone in [62].

Theorem 6.2.1 (The Port–Stone criterion)
Let G be a compactly generated locally compact Abelian group. For any $(\mu_t)_{t \geqslant 0} \in \mathrm{CS}(G)$ the following statements are equivalent:

  (i) $(\mu_t)_{t \geqslant 0}$ *is transient.*
  (ii) $(\mu_t)_{t \geqslant 0}$ *is locally integrable.*

   In view of the proof of Theorem 6.2.1 we now restrict the discussion to *adapted* convolution semigroups $(\mu_t)_{t \geqslant 0} \in \mathrm{CS}(G)$ with

$$(\mu_t)_{t \geqslant 0} \leftrightarrow \psi \leftrightarrow (\varrho_\lambda)_{\lambda > 0} \,.$$

Lemma 6.2.2 (The Chung–Fuchs criterion)
Suppose there exists a relatively compact neighbourhood $U \in \mathfrak{V}_{G^\wedge}(0)$ such that

$$\limsup_{\lambda \to 0} \int_U \mathrm{Re}\, \frac{1}{\lambda + \psi} \,\mathrm{d}\omega_{G^\wedge} < \infty \,.$$

   Then $(\mu_t)_{t \geqslant 0}$ is transient.

Proof.   Since $(\mu_t)_{t \geqslant 0}$ is adapted, we infer from Theorem 5.2.17 that $\psi(\chi) \neq 0$ for $\chi \neq 0$.   Taking into account that

$$\mathrm{Re}\, \frac{1}{\lambda + \psi} \leqslant \frac{\lambda + \mathrm{Re}\,\psi}{|\psi|^2}$$

the hypothesis of the lemma yields

$$\limsup_{\lambda \to 0} \int_C \mathrm{Re}\, \frac{1}{\lambda + \psi} \,\mathrm{d}\omega_{G^\wedge} < \infty$$

for all compact subsets C of $G^\wedge$. It follows that

$$\limsup_{\lambda \to 0} \int_{G^\wedge} \mathrm{Re}\, \frac{1}{\lambda + \psi} \, f * f^\sim \,\mathrm{d}\omega_{G^\wedge} < \infty$$

for all $f \in C^{\mathrm{c}}(G^\wedge)$. But given $g \in C^{\mathrm{c}}_+(G)$ there exists an $f \in C^{\mathrm{c}}(G^\wedge)$ with $|\check{f}|^2 \geqslant g$. This implies that

$$\limsup_{\lambda \to 0} \int_G g \,\mathrm{d}[\tfrac{1}{2}(\varrho_\lambda + \varrho_\lambda^\sim)] \leqslant \limsup_{\lambda \to 0} \int_G |\check{f}|^2 \mathrm{d}[\tfrac{1}{2}(\varrho_\lambda + \varrho_\lambda^\sim)]$$

$$= \limsup_{\lambda \to 0} \int_G f * f^\sim \,\mathrm{Re}\, \frac{1}{\lambda + \psi} \,\mathrm{d}\omega_{G^\wedge} < \infty \,,$$

where the last equality follows from formula (4) in the proof of Theorem 4.4.14 together with Theorem 4.4.16. So we have

$$\limsup_{\lambda \to 0} \int_G g \, d\varrho_\lambda < \infty \, ,$$

hence by monotonicity that the limit

$$\lim_{\lambda \to 0} \int_G g \, d\varrho_\lambda$$

exists and is finite for all $g \in C_+^c(G)$. $\square$

Remark 6.2.3 Applying Theorem 6.1.8 one obtains the converse of the statement in Lemma 6.2.2.

Lemma 6.2.4 *If* $(\mu_t)_{t \geqslant 0}$ *is locally integrable and recurrent, then for every* $f \in C_+^c(G)$ *with* $f \not\equiv 0$ *one has*

(a) $\displaystyle \limsup_{\lambda \to 0} \lambda \varrho_\lambda * \varrho_\lambda^\sim * f * f^\sim(0) = \infty$

and

(b) $\lambda \varrho_\lambda * \varrho_\lambda^\sim * f * f^\sim(0) > 0$ *whenever* $\lambda > 0$.

Proof. Since $(\mu_t)_{t \geqslant 0}$ is assumed to be locally integrable, we have for all relatively compact neighbourhoods $U \in \mathfrak{V}_{G^\wedge}(0)$ that

$$\int_U \frac{\operatorname{Re} \psi}{|\lambda + \psi|^2} \, d\omega_{G^\wedge} \leqslant \int_U \frac{\operatorname{Re} \psi}{|\psi|^2} \, d\omega_{G^\wedge}$$
$$= \int_U \operatorname{Re} \frac{1}{\psi} \, d\omega_{G^\wedge} < \infty \, . \qquad (1)$$

But we also assumed $(\mu_t)_{t \geqslant 0}$ to be recurrent. This implies by Lemma 6.2.2 that

$$\limsup_{\lambda \to 0} \int_U \frac{\lambda + \operatorname{Re} \psi}{|\lambda + \psi|^2} \, d\omega_{G^\wedge} = \limsup_{\lambda \to 0} \int_U \operatorname{Re} \frac{1}{\lambda + \psi} \, d\omega_{G^\wedge} = \infty \, . \qquad (2)$$

Moreover, for $f \in C_+^c(G)$ with $f \not\equiv 0$, and all $\lambda > 0$

$$\lambda \varrho_\lambda * f * (\varrho_\lambda * f)^\sim(0) = \lambda \int_G |\varrho_\lambda * f|^2 \, d\omega_G$$
$$= \lambda \int_{G^\wedge} |\widehat{\varrho_\lambda}|^2 |\hat{f}|^2 \, d\omega_{G^\wedge}$$
$$= \int_{G^\wedge} |\hat{f}|^2 \frac{\lambda}{|\lambda + \psi|^2} \, d\omega_{G^\wedge} > 0 \, .$$

We note that in fact

$$\int_{G^\wedge} |\hat{f}|^2 \frac{\lambda}{|\lambda + \psi|^2} \, \mathrm{d}\omega_{G^\wedge} \geqslant \alpha^2 \int_U \frac{\lambda}{|\lambda + \psi|^2} \, \mathrm{d}\omega_{G^\wedge} \, ,$$

where α is chosen such that $\hat{f}(0) > \alpha > 0$ and U is a relatively compact neighbourhood in $\mathfrak{V}_{G^\wedge}(0)$ within the set $\{\chi \in G^\wedge : |\hat{f}(\chi)| \geqslant \alpha\}$. Together with (1) and (2) this yields

$$\limsup_{\lambda \to 0} \lambda \varrho_\lambda * f * (\varrho_\lambda * f)^\sim(0) = \infty \, .$$

$\square$

Principal Lemma 6.2.5 *Let G be a non-compact second countable locally compact Abelian group, and let $(\mu_t)_{t\geqslant 0}$ be an adapted continuous convolution semigroup in $M^1(G)$. Suppose that*

(α) $(\mu_t)_{t\geqslant 0}$ is locally integrable

and that

(β) there exists an $f \in C_+^c(G)$ with $f \not\equiv 0$ such that

$$\sup_{\lambda \in \,]0,1]} \sigma_\lambda * f * f^\sim(0) < \infty \, ,$$

where

$$\sigma_\lambda := \frac{1}{2}(\varrho_\lambda + \varrho_\lambda^\sim) - \lambda \varrho_\lambda * \varrho_\lambda^\sim \in M_{\mathrm{p}}(G) \, .$$

Then $(\mu_t)_{t\geqslant 0}$ is transient.

Proof. 1. Assuming that $(\mu_t)_{t\geqslant 0}$ is recurrent we infer from Lemma 6.2.4 that with the notation

$$a_\lambda := \lambda \varrho_\lambda * \varrho_\lambda^\sim * f * f^\sim(0)^{-1}$$

for all $\lambda > 0$ we have

$$\liminf_{\lambda \to 0} a_\lambda = 0 \, .$$

This condition implies that $(a_\lambda)_{\lambda \in \Lambda}$ will be bounded for some subset Λ of $]0,1]$ that contains points arbitrarily close to 0. Then $(a_\lambda \varrho_\lambda)_{\lambda \in \Lambda}$ is a τ_v-relatively compact subset of $M_+(G)$.

In order to see this it suffices to prove that

$$\sup_{\lambda \in \,]0,1]} a_\lambda(\varrho_\lambda + \varrho_\lambda^\sim)(K) < \infty$$

for each compact subset K of G. But this will be clear from the following reasoning: We choose a compact subset K of G and a number $c > 0$ such that

$$L := \{x \in G : f * f^\sim(x) > c\} \neq \emptyset.$$

Then, for a suitable sequence $\{x_1, \ldots, x_n\}$ in G $(n \geqslant 1)$ we get

$$K \subset \bigcup_{i=1}^{n} (x_i - L).$$

It follows that

$$(\varrho_\lambda + \varrho_\lambda^\sim)(K) \leqslant \sum_{i=1}^{n} (\varrho_\lambda + \varrho_\lambda^\sim)(x_i - L)$$

$$\leqslant \frac{1}{c} \sum_{i=1}^{n} (\varrho_\lambda + \varrho_\lambda^\sim) * f * f^\sim(x_i).$$

Since $\varrho_\lambda + \varrho_\lambda^\sim \in M_p(G)$, Proposition 4.4.9 implies that $(\varrho_\lambda + \varrho_\lambda^\sim) * f * f^\sim \in \mathrm{CPD}(G)$, and the above sequence of inequalities can be extended to

$$\leqslant \frac{1}{c} \sum_{i=1}^{n} (\varrho_\lambda + \varrho_\lambda^\sim) * f * f^\sim(0)$$

$$= \frac{2n}{c} (\sigma_\lambda * f * f^\sim(0) + a_\lambda^{-1})$$

for all $\lambda \in \,]0, 1]$.

2. The assumptions on G imply that $M_+(G)$ is τ_v-metrizable. Therefore there exists a sequence $(\lambda_k)_{k \geqslant 1}$ in $]0, 1]$ with $\lambda_k \downarrow 0$ such that

$$\lim_{k \to \infty} a_{\lambda_k} = 0$$

and

$$(\tau_v -) \lim_{k \to \infty} a_{\lambda_k} \varrho_{\lambda_k} =: \zeta \in M_+(G).$$

From

$$(\frac{1}{2}(\zeta + \zeta^\sim)) * f * f^\sim(0) = \lim_{k \to \infty} a_{\lambda_k} (\frac{1}{2}(\varrho_{\lambda_k} + \varrho_{\lambda_k}^\sim)) * f * f^\sim(0)$$

$$= \lim_{k \to \infty} a_{\lambda_k} \sigma_{\lambda_k} * f * f^\sim(0) + 1 = 1$$

we conclude that $\zeta \neq 0$.

3. The next aim will be to show that

(a) $\zeta \in M^{\mathrm{sb}}(G)$ and that

(b) $\zeta = \lambda \zeta * \varrho_\lambda$ for all $\lambda > 0$.

This being done Theorem 6.1.12 together with Theorem 5.2.16 (and the adaptation of $(\mu_t)_{t \geqslant 0}$) will imply that

$$\zeta = c\,\omega_G$$

for some $c > 0$.

Now concerning (a) we just note that $\zeta + \zeta^\sim \in M_{\mathrm{p}}(G)$, since $\varrho_\lambda + \varrho_\lambda^\sim \in M_{\mathrm{p}}(G)$, hence $\zeta + \zeta^\sim \in M^{\mathrm{sb}}$ and so also $\zeta \in M^{\mathrm{sb}}$.

As for the proof of (b) we deduce from the resolvent equation that

$$(\lambda - \lambda_k)\varrho_\lambda * a_{\lambda_k}\varrho_{\lambda_k} \leqslant a_{\lambda_k}\varrho_{\lambda_k}$$

for every $\lambda > 0$ and $k \geqslant 1$. Given $\lambda > 0$ we choose $\varepsilon \in]0, \lambda[$ and suppose that $\lambda_k \in]0, \varepsilon[$. Then

$$(\lambda - \varepsilon)\varrho_\lambda * a_{\lambda_k}\varrho_{\lambda_k} \leqslant a_{\lambda_k}\varrho_{\lambda_k}\,,$$

hence

$$(\lambda - \varepsilon)\varrho_\lambda * \zeta \leqslant \zeta\,,$$

and for $\varepsilon \to 0$ we arrive at

$$\lambda \zeta * \varrho_\lambda \leqslant \zeta$$

valid for all $\lambda > 0$.

Suppose now that $\lambda \zeta * \varrho_\lambda \neq \zeta$ for some $\lambda > 0$. Since

$$\sum_{k=0}^{n}(\lambda \varrho_\lambda)^k * (\zeta - \lambda \varrho_\lambda * \zeta) = \zeta - (\lambda \varrho_\lambda)^{n+1} * \zeta \leqslant \zeta\,,$$

the series $\sum_{n \geqslant 1}(\lambda \varrho_\lambda)^n$ norm-converges for $\lambda > 0$, and

$$\sum_{k=0}^{n}\|(\lambda \varrho_\lambda)^k\| = \|\sum_{k=0}^{n}(\lambda \varrho_\lambda)^k\|$$

$$\leqslant \frac{\|\zeta\|}{\|\zeta - \lambda \varrho_\lambda * \zeta\|}\,.$$

But as in the proof of Property 6.1.4.3 we apply the formula

$$\varrho_{\lambda'} = \sum_{n \geqslant 0}(\lambda - \lambda')\varrho_\lambda^{n+1}$$

valid for $\lambda' \in]0, \lambda[$ and obtain

$$\lim_{\lambda' \to 0}\varrho_{\lambda'} = \frac{1}{\lambda}\sum_{n \geqslant 1}(\lambda \varrho_\lambda)^n$$

which implies the transience of $(\mu_t)_{t\geqslant 0}$, hence a contradiction.

4. Now let W be an open, relatively compact subset of G such that $\mathrm{supp}(f * f^\sim) \subset W$. For each $k \geqslant 1$ we consider the convolution semigroup $(\mathrm{e}^{-\lambda_k t} \mu_t^\sim)_{t\geqslant 0} \in \mathrm{CS}(G)$ with potential measure $\varrho_{\lambda_k}^\sim$ and equilibrium measure γ_{λ_k} depending on W, the latter existing by Theorem 6.1.14. For every $k \geqslant 1$ we introduce the measure

$$\nu_{\lambda_k} := a_{\lambda_k}^{-1} \gamma_{\lambda_k} \, ,$$

and we obtain

$$\begin{aligned}
\nu_{\lambda_k}(\overline{W}) a_{\lambda_k} \varrho_{\lambda_k}(\overline{W}) &\leqslant \int_G \int_G \mathbb{1}_{\overline{W}-\overline{W}}(x+y) \, \gamma_{\lambda_k}(\mathrm{d}x) \, \varrho_{\lambda_k}^\sim(\mathrm{d}y) \\
&= (\gamma_{\lambda_k} * \varrho_{\lambda_k}^\sim)(\overline{W} - \overline{W}) \\
&\leqslant \omega_G(\overline{W} - \overline{W}) < \infty \, .
\end{aligned}$$

Since

$$\liminf_{k\to\infty} a_{\lambda_k} \varrho_{\lambda_k}(\overline{W}) > 0 \, ,$$

the sequence $(\nu_{\lambda_k}(\overline{W}))_{k\geqslant 1}$ is bounded, and since $\mathrm{supp}(\gamma_{\lambda_k}) \subset \overline{W}$ for all $k \geqslant 1$, $(\nu_k)_{k\geqslant 1}$ is norm-bounded. Without loss of generality we may assume that $(\nu_{\lambda_k})_{k\geqslant 1}$ τ_v-converges to a measure $\nu \in M_+(G)$. ν is bounded, since $\mathrm{supp}(\nu) \subset \overline{W}$, and $\nu \neq 0$. The latter statement requires a detailed argument. First of all we have that

$$\begin{aligned}
0 < \omega_G(W) = \nu_{\lambda_k} * a_{\lambda_k} \varrho_{\lambda_k}^\sim(W) \\
= a_{\lambda_k} \iint \mathbb{1}_W(x+y) \, \nu_{\lambda_k}(\mathrm{d}x) \, \varrho_{\lambda_k}^\sim(\mathrm{d}y) \\
\leqslant a_{\lambda_k} \int_{W-W} \left(\int_W \mathrm{d}\nu_{\lambda_k} \right) \mathrm{d}\varrho_{\lambda_k}^\sim \\
= a_{\lambda_k} \nu_{\lambda_k}(W) \varrho_{\lambda_k}^\sim(W - W) \, .
\end{aligned}$$

Since W is relatively compact, there exists a function $g \in C_+^c(G)$ such that

$$\mathbb{1}_{W\cup(W-W)} \leqslant g$$

which implies that

$$\limsup_{k\to\infty} \nu_{\lambda_k}(W) \leqslant \lim_{k\to\infty} \nu_{\lambda_k}(g) = \nu(g) \, .$$

Under the assumption that $\nu \equiv 0$ we obtain

$$\lim_{k\to\infty} \nu_{\lambda_k}(W) = 0 \, .$$

From

$$\limsup_{k\to\infty} a_{\lambda_k}\varrho_{\widetilde{\lambda_k}}(W - W) \leqslant \lim_{k\to\infty} a_{\lambda_k}\varrho_{\widetilde{\lambda_k}}(g)$$
$$= \zeta^\sim(g) < \infty$$

we conclude that the sequence $(a_{\lambda_k}\varrho_{\widetilde{\lambda_k}}(W - W))_{k\geqslant 1}$ is bounded, so the product sequence $(a_{\lambda_k}\varrho_{\widetilde{\lambda_k}}(W-W)\nu_{\lambda_k}(W))_{k\geqslant 1}$ tends to 0 and its members are not $\geqslant \omega_G(W) > 0$ which provides a contradiction. Consequently $\nu \in M^{\mathrm{b}}_+(G) \setminus \{0\}$.

5. By the resolvent equation

$$\lim_{k\to\infty} \lambda_k\varrho_{\lambda_k} * \varrho_{\widetilde{\lambda_k}} * (\varepsilon_0 - (\lambda - \lambda_k)\varrho_\lambda) * \nu_{\lambda_k}$$
$$= \lim_{k\to\infty} \lambda_k\varrho_{\widetilde{\lambda_k}} * \nu_{k_\lambda} * \varrho_\lambda ,$$
(3)

and this limit equals 0. For the latter assertion we need to realize that

$$\lambda_k\varrho_{\widetilde{\lambda_k}} * \nu_{\lambda_k} * \varrho_\lambda \leqslant a_{\lambda_k}^{-1}\lambda_k(\varrho_\lambda * \omega_G)$$
$$= a_{\lambda_k}^{-1}\lambda_k\|\varrho_\lambda\|\omega_G$$

and to prove that

$$\lim_{k\to\infty} a_{\lambda_k}^{-1}\lambda_k = 0 .$$

This limit relationship can be shown as follows: from the proof of Lemma 6.2.4 we infer that

$$a_{\lambda_k}^{-1}\lambda_k = \int_{G^\wedge} |\hat{f}|^2 \left(\frac{\lambda_k}{|\lambda_k + \psi|}\right)^2 \mathrm{d}\omega_{G^\wedge}$$

for all $k \geqslant 1$. The integrands are bounded by $|\hat{f}|^2$ and converge (for $k \to \infty$) pointwise to 0 on $G^\wedge \setminus \{0\}$ (since $\psi(\chi) \neq 0$ for $\chi \neq 0$). An application of the Plancherel theorem 4.2.11 (iii) yields that it is also $\omega_{G^\wedge}$-integrable. But then the dominated convergence theorem implies the assertion.

6. We are ready to finish the proof by establishing a contradiction to the boundedness of $(a_{\lambda_k})_{k\geqslant 1}$.

By the choice of f and W according to 4. of this proof we note that

$$\mathrm{supp}(f * f^\sim)^\sim = \mathrm{supp}(f * f^\sim) \subset W .$$

Moreover

$$\|(\lambda - \lambda_k)\varrho_\lambda\| \leqslant 1$$

for all $\lambda \geqslant \lambda_k$ $(k \geqslant 1)$. Corollary 6.1.15 now yields

$$\varrho_{\lambda_k}^{\sim} * (\varepsilon_0 - (\lambda - \lambda_k)\varrho_\lambda) * \nu_{\lambda_k} * f * f^{\sim}(0) \geqslant 0 \tag{4}$$

for $\lambda \geqslant \lambda_k$. But $\sigma_\lambda \in M_{\mathrm{p}}(G)$, since

$$\hat{\sigma}_\lambda = \frac{\operatorname{Re}\psi}{|\lambda + \psi|^2} \geqslant 0\,.$$

Hence

$$\sigma_{\lambda_k} * f * f^{\sim} \leqslant \sigma_{\lambda_k} * f * f^{\sim}(0)$$

and consequently,

$$\begin{aligned}
\sigma_{\lambda_k} &* (\varepsilon_0 - (\lambda - \lambda_k)\varrho_\lambda) * \nu_{\lambda_k} * f * f^{\sim}(0) \\
&= \int_G (\sigma_{\lambda_k} * f * f^{\sim})(-x)\,[(\varepsilon_0 - (\lambda - \lambda_k)\varrho_\lambda) * \nu_{\lambda_k}](\mathrm{d}x) \tag{5} \\
&\leqslant \sigma_{\lambda_k} * f * f^{\sim}(0)\|\nu_{\lambda_k}\|\|\varepsilon_0 - (\lambda - \lambda_k)\varrho_\lambda\|\,.
\end{aligned}$$

But this expression admits a bound M (independent of λ) for all $\lambda_k \leqslant \lambda$, since $(\sigma_{\lambda_k} * f * f^{\sim}(0))_{k \geqslant 1}$ is bounded by assumption and

$$\begin{aligned}
\|\varepsilon_0 - (\lambda - \lambda_k)\varrho_\lambda\| &\leqslant 1 + \|(\lambda - \lambda_k)\varrho_\lambda\| \\
&\leqslant 1 + \|\lambda\varrho_\lambda\| \leqslant 2\,.
\end{aligned}$$

Now we deduce from (3) that

$$\lim_{k \to \infty} \lambda_k * \varrho_{\lambda_k} * \varrho_{\lambda_k}^{\sim} * (\varepsilon_0 - (\lambda - \lambda_k)\varrho_\lambda) * \nu_{\lambda_k} * f * f^{\sim}(0) = 0\,,$$

whence by (5)

$$\limsup_{k \to \infty} (\varrho_{\lambda_k} + \varrho_{\lambda_k}^{\sim}) * (\varepsilon_0 - (\lambda - \lambda_k)\varrho_\lambda) * \nu_{\lambda_k} * f * f^{\sim}(0) \leqslant 2M$$

and thus by (4)

$$\limsup_{k \to \infty} \varrho_{\lambda_k} * (\varepsilon_0 - (\lambda - \lambda_k)\varrho_\lambda) * \nu_{\lambda_k} * f * f^{\sim}(0) \leqslant 2M$$

which together with the resolvent equation implies that

$$\limsup_{k \to \infty} \varrho_\lambda * \nu_{\lambda_k} * f * f^{\sim}(0) \leqslant 2M\,.$$

From $\tau_{\mathrm{v}}\text{-}\lim_{k \to \infty} \nu_{\lambda_k} = \nu$ (part 4. of the proof) and the norm-boundedness of $(\nu_{\lambda_k})_{k \geqslant 1}$ we derive

$$\varrho_\lambda * \nu * f * f^{\sim}(0) \leqslant 2M\,,$$

i.e.

$$\int \nu^{\sim} * f * f^{\sim}\,d\varrho_\lambda \leqslant 2M$$

for all $\lambda > 0$, in particular for $\lambda := \lambda_k$ whenever $k \geqslant 1$. Consequently the sequence

$$\left(a_{\lambda_k}^{-1} \int_G \nu^\sim * f * f^\sim \mathrm{d}(a_{\lambda_k} \varrho_{\lambda_k}) \right)_{k \geqslant 1}$$

is bounded. On the other hand

$$\lim_{k \to \infty} \int_G \nu^\sim * f * f^\sim \, \mathrm{d}(a_{\lambda_k} \varrho_{\lambda_k}) = \int_G \nu^\sim * f * f^\sim \, \mathrm{d}\zeta > 0 \,,$$

the strict positivity following from the facts that $\zeta = c \omega_G$ for $c > 0$ and that

$$\nu^\sim * f * f^\sim \not\equiv 0 \,.$$

But then $(a_{\lambda_k}^{-1})_{k \geqslant 1}$ must be bounded, and this contradicts the assumption made in Part 1. of the proof. The demonstration of the lemma is finally complete. □

Proof of the Port–Stone criterion 6.2.1. By Property 6.1.11.4 it suffices to establish the implication (TC) for an adapted Poisson semigroup $(\mu_t)_{t \geqslant 0}$ determined by a measure $\mu \in M^1(G)$, where G is of the form $\mathbb{R}^d \times \mathbb{Z}^e$ for $a, e \geqslant 1$. For the associated continuous negative definite function of the form $\psi = 1 - \hat{\mu}$ we assume $\operatorname{Re} \frac{1}{\psi}$ to be locally integrable. It will be shown that $(\mu_t)_{t \geqslant 0}$ is transient.

1. The case $d + e \geqslant 3$.

The dual $G^\wedge = \mathbb{R}^d \times \mathbb{T}^e$ of G will be interpreted as the subset $\mathbb{R}^d \times] - \pi, \pi]^e$ of $\mathbb{R}^{d+e}$ equipped with the Euclidean norm $\| \cdot \|$.

At first we show that there exists a constant $c > 0$ such that

$$\operatorname{Re} \psi(y) \geqslant c \| y \|^2 \tag{6}$$

for all y in a relatively compact neighbourhood $U \in \mathfrak{V}_{G^\wedge}(0)$.

In fact, since $G = [\operatorname{supp}(\mu)]^-$, one finds $m := d + e$ linearly independent elements $x_1, \ldots, x_m \in \operatorname{supp}(\mu)$. Let $V := \{ x \in G : \| x \| \leqslant \gamma \}$, where $\gamma := \max\{ \| x_i \| : i = 1, \ldots, m \}$. The quadratic form Q defined by

$$Q(y) := \int_V y(x)^2 \, \mu(\mathrm{d}x)$$

for all $y \in G^\wedge$ is positive definite, hence its smallest eigenvalue τ is strictly positive and

$$Q(y) \geqslant \tau \| y \|^2$$

for all $y \in G^\wedge$. On the other hand

$$\operatorname{Re} \psi(y) = \int_G (1 - \cos y(x)) \, \mu(\mathrm{d}x)$$

$$\leqslant 2 \int_V \sin^2\left(\frac{1}{2} y(x)\right) \mu(\mathrm{d}x)$$

for all $y \in G^\wedge$. But the modulus of the latter integrand being $\geqslant \frac{1}{\pi}|y(x)|$ if $|y(x)| \leqslant \pi$ we obtain that

$$\operatorname{Re} \psi(y) \geqslant \frac{2}{\pi^2} \int_W |y(x)|^2 \, \mu(\mathrm{d}x) \,,$$

for all $y \in G^\wedge$, where

$$W := \{x \in G : \|x\| \leqslant \gamma, \ |y(x)| \leqslant \pi\} \,.$$

If $\|y\| \leqslant \frac{\pi}{\gamma}$ and $\|x\| \leqslant \gamma$ we obtain $|y(x)| \leqslant \pi$. But then

$$\operatorname{Re} \psi(y) \geqslant \frac{2}{\pi^2} \int_V |y(x)|^2 \, \mu(\mathrm{d}x) \geqslant \frac{2}{\pi^2} \tau \|y\|^2$$

for all y in the relatively compact neighbourhood $U := \{u \in G^\wedge : \|u\| \leqslant \frac{\pi}{\gamma}\} \in \mathfrak{V}_{G^\wedge}(0)$. With $c := \frac{2}{\pi^2}\tau$ we arrive at the assertion.

Applying the inequality (6) we now obtain that

$$\int_U \operatorname{Re} \frac{1}{\lambda + \psi} \, \mathrm{d}\lambda^{d+e} \leqslant \int_U \frac{1}{\lambda + \operatorname{Re} \psi} \, \mathrm{d}\lambda^{d+e} \leqslant \frac{1}{c} \int_U \frac{1}{\|y\|^2} \, \lambda^{d+e}(\mathrm{d}y) \,,$$

and the Chung–Fuchs criterion 6.2.2 yields the transience of $(\mu_t)_{t \geqslant 0}$, provided $d + e \geqslant 3$.

2. The case $d + e = 2$. We note that for all $\lambda > 0$

$$\int_U \frac{\lambda}{|\lambda + \psi|^2} \, \mathrm{d}\lambda^2 \leqslant \int_U \frac{\lambda}{|\lambda + \operatorname{Re}\psi|^2} \, \mathrm{d}\lambda^2$$

$$\leqslant \int_U \frac{\lambda}{(\lambda + c\|y\|^2)^2} \, \lambda^2(\mathrm{d}y)$$

$$\leqslant \int_{\mathbb{R}^2} \frac{1}{(1 + c\|y\|^2)^2} \, \lambda^2(\mathrm{d}y) < \infty \,.$$

On the other hand we obtain from the hypothesis that

$$\sup_{\lambda \in \,]0,1]} \int_U \operatorname{Re} \frac{1}{\lambda + \psi} \, \mathrm{d}\lambda^2 = \sup_{\lambda \in \,]0,1]} \int_U \frac{\lambda + \operatorname{Re}\psi}{|\lambda + \psi|^2} \, \mathrm{d}\lambda^2 < \infty$$

which, again by Lemma 6.2.2, implies the assertion.

3. The case $d + e = 1$, i.e. the groups $G = \mathbb{R}$ or $G = \mathbb{Z}$.

From the principal lemma 6.2.5 we infer that it suffices to establish the existence of a function $f \in C_+^c(G)$ with $f \not\equiv 0$ satisfying

$$\sup_{\lambda \in \,]0,1]} \sigma_\lambda * f * f^\sim(0) < \infty.$$

At first we observe that for all $f \in C_+^c(G)$

$$\sigma_\lambda * f * f^\sim(0) = \int_G (\sigma_\lambda * f)\overline{f}\,\mathrm{d}\omega_G = \int_{G^\wedge} \widehat{\sigma_\lambda * f}\,\hat{\overline{f}}\,\mathrm{d}\omega_{G^\wedge}$$

$$= \int_{G^\wedge} \hat{\sigma}_\lambda |\hat{f}|^2\,\mathrm{d}\omega_{G^\wedge} = \int_{G^\wedge} \frac{\mathrm{Re}\,\psi}{|\lambda + \psi|^2}|\hat{f}|^2\,\mathrm{d}\omega_{G^\wedge}$$

$$\leqslant \int_{G^\wedge} \frac{\mathrm{Re}\,\psi}{|\psi|^2}|\hat{f}|\,\mathrm{d}\omega_{G^\wedge} = \int_{G^\wedge} \mathrm{Re}\,\frac{1}{\psi}|\hat{f}|^2\,\mathrm{d}\omega_{G^\wedge}\,.$$

If $G = \mathbb{Z}$, the last integral is finite, since $G^\wedge = \mathbb{T}$ is compact.

Now let $G = \mathbb{R}$. For each relatively compact neighbourhood $U \in \mathfrak{V}_{G^\wedge}(0)$ we have that

$$\sigma_\lambda * f * f^\sim(0) \leqslant \int_{G^\wedge} \mathrm{Re}\,\frac{1}{\psi}|\hat{f}|^2\,\mathrm{d}\omega_{G^\wedge}$$

$$= \int_U \mathrm{Re}\,\frac{1}{\psi}|\hat{f}|^2\,\mathrm{d}\omega_{G^\wedge} + \int_{\complement U} \mathrm{Re}\,\frac{1}{\psi}|\hat{f}|^2\,\mathrm{d}\omega_{G^\wedge}\,.$$

Obviously the integral over U exists. Given a symmetric measure $\nu \in M_+^b(\mathbb{R})$ with compact support such that

$$\nu \leqslant \frac{1}{2}(\mu + \mu^\sim)$$

holds we introduce the desired function f on $\mathbb{R}$ by

$$f(x) := \int_{\mathbb{R}} |x - y|\,\nu(\mathrm{d}y) - \|\nu\|\,|x|$$

whenever $x \in \mathbb{R}$. Since $\mu \neq \varepsilon_0$, we may assume that ν is not a multiple of ε_0, hence $f(0) \neq 0$. It is easily shown that $f \in C_+^c(\mathbb{R})$ and that

$$\hat{f}(x) = \frac{2}{x^2}(\|\nu\| - \hat{\nu}(x))$$

for all $x \in \mathbb{R}^\times$. From the inequalities

$$0 \leqslant \|\nu\| - \hat{\nu}(x) = \int (1 - \cos(xy))\,\nu(\mathrm{d}y)$$

$$\leqslant \int (1 - \cos(xy))\,\mu(\mathrm{d}y) = 1 - \mathrm{Re}\,\hat{\mu}(x)$$

$$= \mathrm{Re}\,\psi(x) \leqslant 2$$

valid for all $x \in \mathbb{R}$ we conclude that

$$\int_{\complement U} \operatorname{Re} \frac{1}{\psi(x)} |\hat{f}(x)|^2 \, \lambda(\mathrm{d}x) \leqslant \int_{\complement U} \frac{4(\operatorname{Re}\psi(x))^2}{x^4 \operatorname{Re}\psi(x)} \, \lambda(\mathrm{d}x)$$

$$\leqslant \int_{\complement U} \frac{8}{x^4} \, \lambda(\mathrm{d}x) < \infty \, .$$

The proof of the Port–Stone criterion 6.2.1 is complete now also for $G = \mathbb{R}$.

<div align="right">□</div>

Examples 6.2.6 We are considering continuous convolution semigroups $(\mu_t)_{t \geqslant 0}$ in $M^1(G)$ with associated continuous negative definite function ψ on $G^\wedge$ admitting a canonical quadruple (in the sense of Theorem 5.4.7) of the form $(0, h, q, 0)$. These semigroups determined by

$$\psi = \mathrm{i}h + q$$

with a homomorphism $h : G^\wedge \to \mathbb{R}$ and a positive quadratic form q on $G^\wedge$ are called (non symmetric) **Gaussian**. The subsequent discussion is designed to study the transience of Gaussian semigroups.

6.2.6.1 Let $(\varepsilon_{x(t)})_{t \geqslant 0}$ be a translation semigroup in $M^1(G)$ as introduced in Example 5.2.7.1. Recall that the mapping $x : \mathbb{R}_+ \to G$ can be extended to a continuous homomorphism $\varphi : \mathbb{R} \to G$. It was shown in Example 5.2.7.1 that $(\varepsilon_{x(t)})_{t \geqslant 0}$ admits a canonical quadruple of the form $(0, h, 0, 0)$.

Now, the semigroup $(\varepsilon_{x(t)})_{t \geqslant 0}$ is transient if and only if $\overline{\varphi(\mathbb{R})}$ is not compact (in G), and in the affirmative case its potential measure can be described as the image $\varphi(\kappa)$ under φ of the potential measure

$$\kappa = \mathbb{1}_{\mathbb{R}_+^\times} \cdot \lambda \in M_+(\mathbb{R})$$

of the translation semigroup $(\varepsilon_t)_{t \geqslant 0}$ in $M^1(\mathbb{R})$. In other words,

$$\varphi(\kappa) = \operatorname{Res}_{\varphi(\mathbb{R}_+^\times)} \omega_{\varphi(\mathbb{R})} \, .$$

In fact, by Appendix A.4.6 a continuous homomorphism $\varphi : \mathbb{R} \to G$ admits the following alternative: either (1) $\overline{\varphi(\mathbb{R})}$ is compact or (2) φ is a homeomorphism onto $\varphi(\mathbb{R})$ and $\varphi(\mathbb{R})$ is a closed subgroup of G. In case (1) $(\varepsilon_{x(t)})_{t \geqslant 0}$ is obviously recurrent (see Discussion 6.1.2.3), in case (2) it is transient, since for every $f \in C_+^c(G)$ the set $\operatorname{supp}(f) \cap \varphi(\mathbb{R})$ is compact, hence for some $t_0 > 0$

$$\int f \, \mathrm{d}\varepsilon_{x(t)} = f \circ \varphi(t) = 0$$

whenever $t \geqslant t_0$.

6.2.6.2 Here we discuss the transience of symmetric Gaussian semigroups $(\nu_t)_{t \geqslant 0}$ admitting a canonical quadruple of the form $(0, 0, q, 0)$. More specifically, we restrict ourselves to the case that $G = \mathbb{R}^d$ $(d \geqslant 1)$ and to the symmetric **Brownian semigroup** $(\nu_t)_{t \geqslant 0}$ given by $\nu_t = n_t \cdot \lambda^d$ with

$$n_t(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{\|x\|^2}{4t}\right)$$

for all $x \in \mathbb{R}^d$ $(t > 0)$.

The following computation shows (by an application of the Schoenberg correspondence theorem 5.2.3) that the function ψ on $\mathbb{R}^d$ defined by

$$\psi(y) := \|y\|^2$$

for all $y \in \mathbb{R}^d$ is the continuous negative definite function associated with $(\nu_t)_{t \geqslant 0}$.

Indeed, for all $y \in \mathbb{R}^d$, $t > 0$ we have

$$\widehat{n_t}(y) = \int_{\mathbb{R}^d} \exp(-i\langle x, y \rangle) \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{\|x\|^2}{4t}\right) \, dx$$

$$= \prod_{k=1}^d \int_{\mathbb{R}} \exp(-ix_k y_k) \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{x_k^2}{4t}\right) \, dx_k$$

$$= \prod_{k=1}^d \exp(-ty_k^2) = \exp(-t\|y\|^2)$$

(with the obvious notation x_k and y_k for the k-th component of the vectors x and y respectively).

Now the Port–Stone criterion 6.2.1 implies that $(\nu_t)_{t \geqslant 0}$ is transient if and only if $d \geqslant 3$.

This result can also be obtained by an explicit computation of the potential measure. In fact, one easily sees that

$$\int_0^\infty n_t(x) \, dt = \begin{cases} \infty & \text{if } d = 1, 2 \text{ for all } x \in \mathbb{R}^d, \\ & \text{and if } d \geqslant 3 \text{ for } x = 0 \\ \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}(d-2)} \|x\|^{2-d} & \text{if } d \geqslant 3 \text{ and } x \in \mathbb{R}^d, \ x \neq 0. \end{cases}$$

Putting for $d \geqslant 3$

$$c_d := \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}(d-2)}$$

and

$$N_d(x) := \begin{cases} \|x\|^{2-d} & \text{if } x \in \mathbb{R}^d, \ x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

we see that the potential measure

$$\kappa_d := c_d N_d \cdot \lambda^d$$

of $(\nu_t)_{t\geqslant 0}$ is a multiple of the classical Newton kernel $N_d \cdot \lambda^d$. We also note that $\kappa_d \in M^\infty(\mathbb{R}^d)$.

6.2.6.3 Symmetric stable semigroups of order $\alpha \in]0,2]$ are given as families $(\mu_t^\alpha)_{t\geqslant 0}$ of measures in $M^1(\mathbb{R}^d)$ $(d \geqslant 1)$ admitting a Fourier representation

$$\widehat{\mu_t^\alpha}(y) = \exp(-t\|y\|^\alpha)$$

for all $y \in \mathbb{R}^d$, $t \geqslant 0$. It can be shown that a symmetric stable semigroup $(\mu_t^\alpha)_{t\geqslant 0}$ of order α is a convolution semigroup subordinated to the symmetric Brownian semigroup $(\nu_t)_{t\geqslant 0}$ defined in Example 6.2.6.2 by means of the one-sided stable semigroup of order $\frac{\alpha}{2}$.

Clearly, the symmetric stable semigroup of order $\alpha = 2$ coincides with the symmetric Brownian semigroup in $M^1(\mathbb{R}^d)$.

Since $\widehat{\mu_t^\alpha} \in L^1(\mathbb{R}^d, \lambda^d)$, an application of the inverse Fourier transform (introduced in the proof to Property 4.2.16.3) yields an integral representation of the λ^d-density m_t^α of μ_t^α as a function in $C_+^0(\mathbb{R}^d)$. In the special cases $\alpha = 1$ and $\alpha = 2$ one may compute m_t^α explicitly. It turns out that

$$m_t^1(x) = \Gamma\left(\frac{d+1}{2}\right) \frac{t}{[\pi(\|x\|^2 + t^2)]^{\frac{d+1}{2}}}$$

whenever $x \in \mathbb{R}^d$, $t \geqslant 0$. If, in addition, $d = 1$ then $(\mu_t^1)_{t\geqslant 0}$ reduces to the **Cauchy semigroup**.

Since

$$(\mu_t^\alpha)_{t\geqslant 0} \leftrightarrow \psi,$$

where $\psi(y) := \|y\|^\alpha$ for all $y \in \mathbb{R}^d$, the λ^d-local integrability of $\frac{1}{\psi}$ which occurs exactly in the cases

$$\begin{cases} d = 1 \ \text{ and } \ \alpha \in]0,1[, \\ d = 2 \ \text{ and } \ \alpha \in]0,2[, \\ d \geqslant 3 \ \text{ and } \ \alpha \in]0,2] \end{cases}$$

implies the transience of $(\mu_t^\alpha)_{t \geqslant 0}$ exactly for these choices of d and α.

For $d = 1$ and $\alpha \in [1, 2]$ as well as for $d = 2$ and $\alpha = 2$ the convolution semigroup $(\mu_t^\alpha)_{t \geqslant 0}$ is recurrent.

In the cases of transience the potential measure $\kappa_{d,\alpha}$ of $(\mu_t^\alpha)_{t \geqslant 0}$ appears to be the **Riesz kernel** of order α having a λ^d-density

$$x \mapsto \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{2^\alpha \pi^{\frac{d}{2}} \Gamma(\frac{\alpha}{2})} \|x\|^{\alpha - d}.$$

For the computation of this density one employs the construction of $(\mu_t^\alpha)_{t \geqslant 0}$ by means of subordination.

6.2.6.4 Let $(\mu_t)_{t \geqslant 0}$ denote the heat semigroup in $M^1(\mathbb{R}^d \times \mathbb{R})$ defined by

$$\mu_t := \nu_t \otimes \varepsilon_t$$

for all $t \geqslant 0$, where $(\nu_t)_{t \geqslant 0}$ and $(\varepsilon_t)_{t \geqslant 0}$ denote the Brownian and the translation semigroups discussed in Examples 6.2.6.2 and 6.2.6.1 respectively. One has

$$(\mu_t)_{t \geqslant 0} \leftrightarrow \psi$$

with

$$\psi(y, s) := \|y\|^2 + \mathrm{i}s$$

whenever $(y, s) \in \mathbb{R}^d \times \mathbb{R}$. Clearly, $(\mu_t)_{t \geqslant 0}$ is locally integrable, and hence the Port–Stone criterion 6.2.1 implies the transience of $(\mu_t)_{t \geqslant 0}$.

The transience of $(\mu_t)_{t \geqslant 0}$ can also be obtained from an explicit computation of the potential measure.

In fact, let $(\mu_t)_{t \geqslant 0} \leftrightarrow (\varrho_\lambda)_{\lambda > 0}$. Then for each $g \in C^c(\mathbb{R}^d \times \mathbb{R})$ we have that

$$\varrho_\lambda(g) = \int_0^\infty \mathrm{e}^{-\lambda s} \mu_s(g) \, \mathrm{d}s$$

$$= \int_0^\infty \mathrm{e}^{-\lambda s} \left[\int_{\mathbb{R}^d} g(x, s) n_s(x) \, \mathrm{d}x \right] \mathrm{d}s \qquad (\lambda > 0).$$

But the function k on $\mathbb{R}^d \times \mathbb{R}$ given by

$$k(x, s) := \begin{cases} n_s(x) & \text{if } x \in \mathbb{R}^d, \ s \in \mathbb{R}_+^\times \\ 0 & \text{if } x \in \mathbb{R}^d, \ s \notin \mathbb{R}_+^\times \end{cases}$$

is locally $(\lambda^{d+1}$-)integrable, hence $(\mu_t)_{t \geqslant 0}$ is transient with potential measure

$$\kappa = k \cdot \lambda^{d+1}.$$

Moreover, $\kappa \in M^\infty(\mathbb{R}^d \times \mathbb{R})$.

6.2.6.5 Let $G := \mathbb{Z}$ and let $(\mu_t)_{t \geqslant 0}$ denote the Poisson semigroup determined by the measure

$$\mu := \sum_{n \geqslant 2} \frac{a}{n^2 \log n} \varepsilon_n \in M^1(G)$$

with a suitably chosen $a > 0$.

Clearly,

$$\kappa := \sum_{n \geqslant 0} \mu^n$$

exists, since $\mathrm{supp}(\mu^n) \subset \{2n, 2n+1, \dots\}$ $(n \in \mathbb{N})$. Observe that $\kappa(n) = 0$ for all $n < 0$. Thus $(\mu_t)_{t \geqslant 0}$ is transient. Moreover it follows from Spitzer's book [79] that $\kappa \in M^\infty(G)$.

On the other hand, $(\mu_t)_{t \geqslant 0}$ being determined by μ is associated with the negative definite function ψ on $\mathbb{T}$ given by

$$\psi(\vartheta) := 1 - \hat{\mu}(\vartheta) = \sum_{n \geqslant 2} \frac{a}{n^2 \log n}(1 - \cos(n\vartheta)) + \mathrm{i} \sum_{n \geqslant 2} \frac{a}{n^2 \log n} \sin(n\vartheta)$$

whenever $\vartheta \in \mathbb{R}$ (after suitable identification of functions on $\mathbb{Z}^\wedge = \mathbb{T}$ and on $\mathbb{R}$). A result in Zygmund's book [102] yields

$$\psi(\vartheta) \sim \mathrm{i}a\vartheta \log |\log \vartheta|$$

for $\vartheta \downarrow 0$, hence $\frac{1}{\psi}$ is not integrable over any neighbourhood of 0.

Altogether we have seen that $(\mu_t)_{t \geqslant 0}$ is transient, hence $\mathrm{Re}\,\frac{1}{\psi}$ is locally integrable by the transience criterion 6.1.8, but $\frac{1}{\psi}$ is not locally integrable.

6.3 Recurrent random walks

Until now we have studied convolution semigroups on an arbitrary locally compact Abelian group G without particular reference to probability theory. In the case of a second countable group G this reference can be easily established. Given a convolution semigroup $(\mu_t)_{t\in\mathbb{R}_+}$ in $M^1(G)$ there exists by the Kolmogorov consistency theorem a translation invariant Markov process in G whose transition semigroup $(P_t)_{t\in\mathbb{R}_+}$ (of Markov kernels P_t on $K \times \mathfrak{B}(G)$) is related to $(\mu_t)_{t\in\mathbb{R}_+}$ by

$$P_t(x, B) = \mu_t * \varepsilon_x(B) \tag{1}$$

for all $(x, B) \in G \times \mathfrak{B}(G)$, $t \in \mathbb{R}_+$. More precisely, there is a one-to-one correspondence between convolution semigroups $(\mu_t)_{t\in\mathbb{R}_+}$ in $M^1(G)$ and stationary independent increment processes with transition semigroup $(P_t)_{t\in\mathbb{R}_+}$ given by (1).

In the present section we shall study stationary independent increment processes in G with transition function $(P_n)_{n\in\mathbb{Z}_+}$, where $P_n := P^n$ for some transition kernel P of the form

$$P(x, B) := \mu * \varepsilon_x(B)$$

for all $(x, B) \in G \times \mathfrak{B}(G)$ with a measure μ in $M^1(G)$. Such processes are called **random walks** in G **with law** μ.

A sketch of the construction of such random walks now follows. Given P and a measure $\nu \in M^1(G)$ there exists a measurable space $(\Omega, \mathfrak{A})$ with $\Omega := G^{\mathbb{Z}_+}$ and $\mathfrak{A} := \mathfrak{B}(G)^{\otimes \mathbb{Z}_+}$ and a measure P^ν on $(\Omega, \mathfrak{A})$ such that the sequence $(X_n)_{n\in\mathbb{Z}_+}$ of projections $X_n : G^{\mathbb{Z}_+} \to G$ (generating the σ-algebra $\mathfrak{A}$) forms a Markov chain in G with transition kernel P and starting measure ν. This statement can be made precise by the defining properties

$$\mathsf{E}^\nu(f \circ X_n \mid \mathfrak{A}_n) = P(f \circ X_n) \quad [\mathsf{P}^\nu]$$

for all bounded measurable functions f on G, where $\mathfrak{A}_n := \sigma(\{X_0, X_1, \dots, X_n\})$ for every $n \in \mathbb{Z}_+$, and

$$\mathsf{P}^\nu([X_0 \in B]) = \nu(B)$$

for all $B \in \mathfrak{B}(G)$.

In the special case $\nu := \varepsilon_x$ for $x \in G$ we shall write E^x and P^x in place of E^ν and P^ν respectively.

Introducing the shift operator θ on Ω by

$$\theta((x_0, x_1, \dots, x_n, \dots)) := (x_1, x_2, \dots, x_{n+1}, \dots)$$

for all $(x_0, x_1, \ldots, x_n, \ldots) \in \Omega$ one obtains the *Markov property* of the chain $(X_n)_{n \in \mathbb{Z}_+}$ in the form

$$\mathsf{E}^\nu(Z \circ \theta^n \mid \mathfrak{A}_n) = \mathsf{E}^{X_n}(Z) \quad [\mathsf{P}^\nu]$$

for every $\mathfrak{A}$-measurable function $Z \geqslant 0$ on Ω, $n \in \mathbb{Z}_+$.

Moreover, it turns out that $(X_n)_{n \in \mathbb{Z}_+}$ enjoys the *strong Markov property* : Given any stopping time τ for $(X_n)_{n \in \mathbb{Z}_+}$ then for every $\mathfrak{A}$-measurable function $Z \geqslant 0$ on Ω

$$\mathsf{E}^\nu(Z \circ \theta_\tau \mid \mathcal{F}_\tau) = \mathsf{E}^{X_\tau}(Z) \quad [\mathsf{P}^\nu],$$

where θ_τ, $\mathcal{F}_\tau$ and X_τ denote the τ-shift, τ-past and the τ-stopped chain respectively.

The random walk in G with law $\mu \in M^1(G)$ will be abbreviated by $X(\mu)$ if no other specification is needed.

It is easy to see that the random walk $X(\mu)$ constructed above has independent increments $Z_n := X_n - X_{n-1}$ $(n \geqslant 1)$ (with respect to P^ν) and that these increments are identically distributed with

$$(\mathsf{P}^\nu)_{Z_n} = \mu$$

for all $n \geqslant 1$.

For a given random walk $X(\mu)$ with law $\mu \in M^1(G)$ we introduce the notation

$$S(\mu) := \langle \mathrm{supp}(\mu) \rangle^-$$

and

$$G(\mu) := [S(\mu)]^- .$$

Observation 6.3.1 $x \in S(\mu)$ if and only if for each $V \in \mathfrak{V}_G(x)$ there exists an $n \geqslant 1$ such that

$$\mu^n(V) = \mathsf{P}^0([X_n \in V]) > 0 .$$

Since G is assumed to be second countable, this implies that

$$\mathsf{P}^0 \left(\bigcap_{n \geqslant 1} [X_n \in S(\mu)] \right) = 1$$

and, after translation in G,

$$\mathsf{P}^x \left(\bigcap_{n \geqslant 0} [X_n \in G(\mu)] \right) = 1$$

for all $x \in G(\mu)$. This means that $G(\mu)$ is an **absorbing set** of $X(\mu)$, hence we may assume without loss of generality that

$$G(\mu) = G.$$

In terms of Fourier transforms this hypothesis can be rewritten as

$$\{\chi \in G^\wedge : \hat{\mu}(\chi) = 1\} = \{0\}.$$

Indeed, if $X(\mu)$ is adapted in the sense of the hypothesis, then 1 being an extremal point of the unit disk, $\hat{\mu}(\chi) = 1$ implies that $\chi(x) = 1$ for μ-a.a. $x \in G$, hence for all $x \in G(\mu)$, and thus (by adaption of μ) for all $x \in G$ (without exception). Thus χ is the unit character 0 in $G^\wedge$.

In the following we shall look at sets of the form

$$R(B) := \limsup_{n \to \infty}[X_n \in B]$$

$$:= [X_n \in B \text{ for infinitely many } n \geqslant 0],$$

where $B \in \mathfrak{B}(G)$.

Definition 6.3.2 $x \in G$ is said to be **recurrent** if for every $V \in \mathfrak{V}_G(x)$ one has

$$\mathsf{P}^0(R(V)) = 1.$$

$x \in G$ is called **transient** if x is not recurrent.

Theorem 6.3.3 (Dichotomy)
For each adapted random walk $X(\mu)$ in G one has the following alternative: Either

(1) *every element of G is recurrent*

or

(2) *every element of G is transient.*

Proof. Let R_μ denote the set of recurrent elements (for $X(\mu)$). Plainly, R_μ is a closed set, and hence it suffices to show that

$$R_\mu - S(\mu) \subset R_\mu.$$

In fact, if this implication holds true, then $R_\mu \subset S(\mu)$ and consequently

$$R_\mu - R_\mu \subset R_\mu.$$

So, if $R_\mu \neq \emptyset$, then R_μ is a closed subgroup of G. But $R_\mu \supset -S(\mu)$ implies that $R_\mu = G$, since $X(\mu)$ is adapted.

Now, let $x \in \mathrm{R}_\mu$ and $y \in S(\mu)$. We shall show that $x - y \in \mathrm{R}_\mu$. Let U be a neighbourhood $\in \mathfrak{V}_G(x - y)$. Clearly, $U + (y - x) \in \mathfrak{V}_G(0)$, and there exists a neighbourhood $V \in \mathfrak{V}_G(0)$ such that $V - V \subset U + (y - x)$. On the other hand, $V + y \in \mathfrak{V}_G(y)$, hence there exists a $k \in \mathbb{Z}_+$ satisfying

$$\mathsf{P}^0([X_k \in V + y]) > 0\,,$$

since $y \in S(\mu)$. Let

$$A := [X_k \in V + y]\,.$$

If $\omega \in A$, the relation $X_{n+k}(\omega) \in V + x$ implies that

$$(X_{n+k} - X_k)(\omega) \in V + x - (V + y) \subset U\,.$$

As x is recurrent, for P^0-a.a. $\omega \in A$ there exists an infinity of integers $n \in \mathbb{Z}_+$ such that $X_{n+k}(\omega) \in V + x$, therefore we obtain

$$\mathsf{P}^0(\limsup_{n \to \infty}[X_{n+k} - X_k \in U] \cap A) = \mathsf{P}^0(A)\,.$$

The sequence $(X_{n+k} - X_k)_{n \in \mathbb{Z}_+}$ having the same distribution as $(X_n)_{n \in \mathbb{Z}_+}$ is independent of A, therefore

$$\mathsf{P}^0(\limsup_{n \to \infty}[X_n \in U]) = 1$$

and $x - y \in \mathrm{R}_\mu$. $\hfill\square$

The dichotomy theorem enables us to make the following

Definition 6.3.4 A random walk $X(\mu)$ with law $\mu \in M^1(G)$ is said to be **recurrent** if it satisfies (1) of the theorem. Otherwise $X(\mu)$ is said to be **transient**.

A locally compact Abelian group G is called **recurrent** provided there exists an adapted recurrent random walk on it, and **transient** otherwise.

We recall the notion of **(first) return time** of $X(\mu)$ into a set $B \in \mathfrak{B}(G)$ defined by

$$\mathrm{R}_B := \begin{cases} \inf\{n \in \mathbb{N} : X_n \in B\} & \text{if } \{n \in \mathbb{N} : X_n \in B\} \neq \emptyset \\ \infty & \text{otherwise.} \end{cases}$$

It is known that R_B is a stopping time for $X(\mu)$ (with respect to the canonical filtration $(\mathfrak{A}_n)_{n \in \mathbb{Z}_+}$).

Theorem 6.3.5 *An adapted random walk* $X(\mu)$ *with law* $\mu \in M^1(G)$ *is recurrent if and only if for each neighbourhood* $V \in \mathfrak{V}_G(0)$ *one has*

$$\mathsf{P}^0([R_V < \infty]) = 1\,.$$

Proof. It suffices to prove that the condition is sufficient for $X(\mu)$ to be recurrent. Let V be a symmetric neighbourhood $\in \mathfrak{V}_G(0)$ and let W be a compact symmetric neighbourhood $\in \mathfrak{V}_G(0)$ such that $W \subset V$. Clearly, there exists an open neighbourhood $U \in \mathfrak{V}_G(0)$ satisfying

$$W + \complement V = -W + \complement V \subset \complement U .$$

It follows that

$$[X_{m+n} \notin V] \cap [X_m \in W] \subset [X_{m+n} - X_m \notin U] ,$$

hence that

$$\mathsf{P}^0 \left([X_m \in W] \cap \bigcap_{n \geqslant 1} [X_{m+n} \notin V] \right)$$

$$\leqslant \mathsf{P}^0 \left([X_m \in W] \cap \bigcap_{n \geqslant 1} [X_{m+n} - X_m \notin U] \right)$$

$$= \mathsf{P}^0([X_m \in W]) \, \mathsf{P}^0([R_U = \infty]) ,$$

since $\mathsf{P}^0_{X_{m+n} - X_m} = \mathsf{P}^0_{X_n}$, and $X_{m+n} - X_m$ is independent of X_m $(m, n \in \mathbb{Z}_+)$. But by hypothesis

$$\mathsf{P}^0([X_m \in W] \cap \bigcap_{n \geqslant 1} [X_{m+n} \notin V]) = 0$$

for all $m \in \mathbb{Z}_+$. Let therefore $(W_r)_{r \geqslant 1}$ denote an increasing sequence of compact symmetric neighbourhoods $W_r \in \mathfrak{V}_G(0)$ such that $\bigcup_{r \geqslant 1} W_r = V$.

Since

$$[X_m \in V] = \lim_{r \to \infty} [X_m \in W_r] ,$$

one has

$$\mathsf{P}^0 \left([X_m \in V] \cap \bigcap_{n \geqslant 1} [X_{m+n} \notin V] \right) = 0$$

for all $m \in \mathbb{Z}_+$. Let now

$$T_V := \begin{cases} \sup\{m \geqslant 0 : X_m \in V\} & \text{if } \{m \geqslant 0 : X_m \in V\} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

be the *last killing time* of V. Then

$$\mathsf{P}^0([T_V < \infty]) = \sum_{m \geqslant 0} \mathsf{P}^0([T_V = m])$$

$$= \sum_{m \geqslant 0} \mathsf{P}^0 \left([X_m \in V] \cap \bigcap_{n \geqslant 1} [X_{m+n} \notin V] \right) = 0 ,$$

hence $\mathsf{P}^0(R(V)) = 1$ which implies that 0 and therefore $X(\mu)$ is recurrent. $\qquad \square$

If $X(\mu)$ is a random walk in G with law $\mu \in M^1(G)$ admitting a transition kernel P we introduce the **potential kernel** K of $X(\mu)$ by

$$Kf(x) := \sum_{n \geqslant 0} P^n f(x) = \mathsf{E}^x(\sum_{n \geqslant 0} f \circ X_n)$$

for all measurable functions $\geqslant 0$ on G and all $x \in G$.

Properties 6.3.6 of the potential kernel.

6.3.6.1 K satisfies the **maximum principle** in the sense that

$$\sup_{x \in G} Kf(x) = \sup\{Kf(y) : y \in \mathrm{supp}(f)\}$$

for all measurable functions f on G .

6.3.6.2 *For each measurable function* $f \geqslant 0$ *on* G *one has*

$$f + PKf = f + KPf \,,$$

hence

6.3.6.3 *the **Poisson equation***

$$(I - P)Kf = f$$

is fulfilled whenever Kf is finite.

6.3.6.4 *For all* $x \in G$ *one has the commutation relationship*

$$T_x P = P T_x \,.$$

The proofs of these properties are straightforward except that of Property 6.3.6.1. Let $x \in G$ and let R^x denote the return time into $\mathrm{supp}(f)$ of the random walk $X(\mu)$ starting at x . (Here we refer to the sequence $(S_n^x)_{n \geqslant 0}$ of shifted sums

$$S_n^x := \sum_{k=1}^n X_k + x$$

$(n \geqslant 1)$ and $S_0^x := x$.) One has

$$Kf(x) = \mathsf{E}^x \left(\sum_{n \geqslant 0} f \circ X_n \right)$$

$$= \mathsf{E}^x \left(\sum_{n \geqslant R^x} f \circ X_n ; R^x < \infty \right)$$

$$= \mathsf{E}^x \left(\sum_{n \geqslant 0} f \circ X_n \circ \theta_{R^x} ; R^x < \infty \right)$$

$$= \mathsf{E}^x \left(\mathsf{E}^{X_{R^x}} \left(\sum_{n \geqslant 0} f \circ X_n \right) ; R_x < \infty \right)$$

(by the strong Markov property of $X(\mu)$)

$$= \mathsf{E}^x(Kf \circ X_{R^x}; R^x < \infty),$$

hence the assertion by taking the supremum over all $x \in G$.

For each pair $(x, B) \in G \times \mathfrak{B}(G)$ we set

$$K(x, B) := K\mathbb{1}_B(x)$$
$$= \kappa * \varepsilon_x(B),$$

where

$$\kappa := \sum_{n \geqslant 0} \mu^n$$

is the elementary kernel determined by $\mu \in M^1(G)$ in the sense of Example 6.1.3.

Theorem 6.3.7 *Let $X(\mu)$ be an adapted random walk in G with law $\mu \in M^1(G)$.*

Then

(i) *$X(\mu)$ is recurrent if and only if $\kappa(U) = \infty$ for every non-empty open subset U of G.*
(ii) *$X(\mu)$ is transient if and only if $\kappa(U) < \infty$ for every relatively compact open subset U of G.*

Proof.   It suffices to show that $X(\mu)$ is recurrent provided $\kappa(U) = \infty$ for each relatively compact open neighbourhood U from a neighbourhood base of 0.

We are retaining the notation of the proof Theorem 6.3.5.

First of all we justify that without loss of generality we can choose U to be a symmetric relatively compact neighbourhood $\in \mathfrak{V}_G(0)$. In fact, let

$$\kappa_N := \sum_{n=0}^N \mu^n \quad \text{for } N \geqslant 1.$$

We have

$$\kappa_N(U) = \mathsf{E}^0 \left(\sum_{n=0}^N \mathbb{1}_U \circ X_n \right)$$

$$= 1 + \mathsf{E}^0 \left(\sum_{n=R_U}^N \mathbb{1}_U \circ X_n \right),$$

hence

$$\kappa_N(U) \leqslant 1 + \mathsf{E}^0 \left(\sum_{n=0}^{N} \mathbb{1}_U \circ X_n \circ \theta_{R_U} \right)$$

$$= 1 + \mathsf{E}^0 \left(\mathsf{E}^{X_{R_U}} \left(\sum_{n \geqslant 0} \mathbb{1}_U \circ X_n \right) ; R_U < \infty \right),$$

which follows from the strong Markov property of $X(\mu)$, and by Property 6.3.6.4 we obtain that

$$\kappa_N(U) \leqslant \mathsf{E}^0(\kappa_N(U - X_{R_U}); R_U < \infty)$$
$$\leqslant \kappa_N(U - U),$$

where $U - U$ is a symmetric relatively compact open neighbourhood $\in \mathfrak{V}_G(0)$. Letting $N \to \infty$, one deduces that

$$\kappa(U - U) = \infty.$$

Now let U be a symmetric relatively compact open neighbourhood in $\mathfrak{V}_G(0)$ such that

$$\kappa(U) = \infty.$$

One has

$$1 \geqslant \mathsf{P}^0([T_U < \infty])$$

$$= \sum_{m \geqslant 0} \mathsf{P}^0 \left([X_m \in U] \cap \bigcap_{n \geqslant 1} [X_{m+n} \notin U] \right)$$

and, since $y - x \notin 2U$ implies $y \notin U$ (for $x \in U$),

$$1 \geqslant \sum_{m \geqslant 0} \mathsf{P}^0 \left([X_m \in U] \cap \bigcap_{n \geqslant 1} [X_{m+n} - X_m \notin 2U] \right).$$

It follows that

$$1 \geqslant \kappa(U) \, \mathsf{P}^0 \, [R_{2U} = \infty],$$

and since $\kappa(U) = \infty$ by assumption, that

$$\mathsf{P}^0([R_{2U} = \infty]) = 0,$$

hence that

$$\mathsf{P}^0([R_{2U} < \infty]) = 1,$$

which by Theorem 6.3.5 completes the proof. $\qquad\square$

Corollary 6.3.8 *If* $X(\mu)$ *is transient, then*

(i) K *is a* proper kernel *in the sense that* $G = \bigcup_{n \geqslant 1} G_n$ *for an increasing sequence* $(G_n)_{n \geqslant 1}$ *in* G *such that* $K(\cdot, G_n)$ *is a bounded function for all* $n \geqslant 1$.

(ii) $K(x, \cdot) \in M_+(G)$ *for all* $x \in G$.

(iii) *The set* $\{K(x, \cdot) : x \in G\}$ *is* τ_v-*relatively compact in* $M_+(G)$.

Proof.   From the theorem we infer that there exists an open $U \in \mathfrak{V}_G(0)$ such that $\kappa(U) < \infty$. Let V be an open neighbourhood $\in \mathfrak{V}_G(0)$ satisfying $V - V \subset U$. Then we obtain for each $x \in V$ that

$$K(x, V) = \kappa(V - x) \leqslant \kappa(V - V) \leqslant \kappa(U)$$

and by the maximum principle 6.3.6.1 that

$$K(x, V) \leqslant \kappa(U)$$

for all $x \in G$. For every compact subset C of G there is a sequence $\{x_1, \ldots, x_n\}$ in C such that the sequence $\{V - x_i : i = 1, \ldots, n\}$ forms an open covering of C. But then

$$K(x, C) \leqslant \sum_{i=1}^n K(x, V - x_i)$$
$$= \sum_{i=1}^n K(x + x_i, V)$$
$$\leqslant n\kappa(V)$$

for all $x \in G$ which implies all the assertions (i) to (iii) at one time. $\square$

Examples 6.3.9 of recurrent random walks.

6.3.9.1 If G is compact, every (adapted) random walk $X(\mu)$ in G is recurrent.

In fact, the potential kernel (measure) κ of $X(\mu)$ satisfies

$$\kappa(G) = \mathsf{E}^0 \left(\sum_{n \geqslant 0} \mathbb{1}_G \circ X_n \right) = \infty$$

for the (relatively) compact open (sub)set G.

6.3.9.2 For the **Bernoulli walk** $X(\mu)$ in $\mathbb{Z}$ with determining measure

$$\mu := \frac{1}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_{-1} \in M^1(\mathbb{Z})$$

(arising as the common law of independent $\mathbb{Z}$-random variables Y_k ($k \geqslant 1$) leading to partial sums $X_n := \sum_{k=1}^{n} Y_k$ ($n \geqslant 1$)) one computes

$$\kappa(\{0\}) = \sum_{n \geqslant 0} \mu^n(\{0\}) = \sum_{n \geqslant 0} \mathsf{P}([X_{2n} = 0]) = \sum_{n \geqslant 0} \binom{2n}{n} \frac{1}{2^{2n}} \,,$$

where

$$\binom{2n}{n} \frac{1}{2^{2n}} = \frac{2n!}{n!n!} \cdot \frac{1}{2^{2n}} \sim \frac{1}{\sqrt{\pi n}}$$

(by Stirling's formula), and obtains that $\kappa(\{0\}) = \infty$ which says that $\mathbb{Z}$ is a recurrent group.

We note (and shall show later) that the random walk $X(\mu)$ in $\mathbb{Z}$ with determining measure

$$\mu := p\varepsilon_1 + q\varepsilon_{-1} \,,$$

where $p, q \geqslant 0$, $p + q = 1$ but $p \neq q \neq \frac{1}{2}$, is transient.

6.3.9.3 Considering the random walk $X(\mu)$ in $\mathbb{Z}^2$ with determining measure

$$\mu := \frac{1}{4}(\varepsilon_{(1,0)} + \varepsilon_{(-1,0)} + \varepsilon_{(0,1)} + \varepsilon_{(0,-1)}) \in M^1(\mathbb{Z}^2)$$

one obtains that

$$\kappa(\{(0,0)\}) = \sum_{n \geqslant 0} \mathsf{P}([X_{2n} = (0,0)])$$

$$= \sum_{n \geqslant 0} \sum_{i+j=n} \frac{2n!}{i!i!j!j!} \left(\frac{1}{4}\right)^{2n} \,,$$

where the inner sums

$$u_n = \left(\frac{1}{4}\right)^{2n} \binom{2n}{n} \sum_{i+j=n} \binom{n}{i}\binom{n}{n-i}$$

$$= \left(\frac{1}{4}\right)^{2n} \binom{2n}{n}^2$$

are asymptotically equal to $\frac{1}{\pi n}$ as $n \to \infty$, hence that $\kappa(\{(0,0)\}) = \infty$. Consequently $\mathbb{Z}^2$ is recurrent.

We are now going to describe the class $\mathcal{R}$ of all recurrent (locally compact Abelian) groups. By a method similar to that applied in the proof of the Chung–Fuchs criterion 6.2.2 we shall at first characterize transient random walks in G.

Theorem 6.3.10 *For any random walk $X(\mu)$ with law $\mu \in M^1(G)$ the following statements are equivalent:*

(i) $X(\mu)$ *is transient.*

(ii) *There exists a neighbourhood* $W \in \mathfrak{V}_{G^{\wedge}}(0)$ *such that*

$$\limsup_{t \uparrow 1} \int_W \operatorname{Re} \frac{1}{1 - t\hat{\mu}} \, d\omega_{G^{\wedge}} < \infty \,.$$

Proof.   (ii) $\Rightarrow$ (i).   For any compact symmetric neighbourhood $V \in \mathfrak{V}_{G^{\wedge}}(0)$ we have

$$\left(\int_V \overline{\chi(x)} \omega_{G^{\wedge}}(d\chi) \right)^2 = \int_{G^{\wedge}} \overline{\chi(x)} \left(\int \mathbb{1}_V(\varrho) \mathbb{1}_V(\chi - \varrho) \, \omega_{G^{\wedge}}(d\varrho) \right) \omega_{G^{\wedge}}(d\chi)$$

$$= \int_{G^{\wedge}} \overline{\chi(x)} \omega_{G^{\wedge}}(V \cap (V + \chi)) \, \omega_{G^{\wedge}}(d\chi) \,,$$

whenever $x \in G$. Integration with respect to μ yields

$$\int_{G^{\wedge}} \hat{\mu}(\chi) \omega_{G^{\wedge}}(V \cap (V + \chi)) \, \omega_{G^{\wedge}}(d\chi)$$

$$= \int_G \left(\int_V \overline{\chi(x)} \, \omega_{G^{\wedge}}(d\chi) \right)^2 \mu(dx) \,.$$

This identity holding true for the μ^n instead of μ, we may multiply by t^n for $t \in [0, 1[$ and sum over n in order to obtain

$$\int_{G^{\wedge}} \frac{1}{1 - t\hat{\mu}(\chi)} \omega_{G^{\wedge}}(V \cap (V + \chi)) \, \omega_{G^{\wedge}}(d\chi)$$

$$= \sum_{n \geqslant 0} t^n \int_G \left(\int_V \overline{\chi(x)} \, \omega_{G^{\wedge}}(d\chi) \right)^2 \mu^n(dx) \,.$$

Now let W be a symmetric compact neighbourhood $\in \mathfrak{V}_{G^{\wedge}}(0)$ and choose $V \in \mathfrak{V}_{G^{\wedge}}(0)$ such that $2V \subset W$. The function

$$\chi \mapsto \omega_{G^{\wedge}}(V \cap (V + \chi))$$

on $G^{\wedge}$ is zero on $\complement W$, hence the left-hand side of the above equality is a real number

$$\leqslant \omega_{G^{\wedge}}(V) \int_W \operatorname{Re} \frac{1}{1 - t\hat{\mu}} \, d\omega_{G^{\wedge}} \,.$$

Since the function

$$x \mapsto \left(\int_V \overline{\chi}(x) \, d\omega_{G^{\wedge}} \right)^2$$

on G is continuous and takes the value $\omega_{G^{\wedge}}(V)^2$ at $x = 0$, it is $\geqslant \frac{1}{2}(\omega_{G^{\wedge}}(V))^2$ on some neighbourhood $U \in \mathfrak{V}_G(0)$. As a consequence we obtain

$$2 \int_W \operatorname{Re} \frac{1}{1 - t\hat{\mu}} \, d\omega_{G^{\wedge}} \geqslant \omega_{G^{\wedge}}(V) \sum_{n \geqslant 0} t^n \, \mathsf{P}^0([X_n \in U]) \,.$$

If, now, $X(\mu)$ is recurrent, then

$$\lim_{t \uparrow 1} \int_W \operatorname{Re} \frac{1}{1 - t\hat{\mu}} \, \mathrm{d}\omega_{G^\wedge} = \infty,$$

and the implication (i) $\Rightarrow$ (ii) has been proved.

(i) $\Rightarrow$ (ii). Let $X(\mu)$ be transient, and let V be a compact symmetric neighbourhood $\in \mathfrak{V}_G(0)$. The function

$$y \mapsto (\mathbb{1}_V * \mathbb{1}_V * \mu)(y) = \int_G \omega_G((V + y) \cap (V + x)) \, \mu(\mathrm{d}x)$$

is an element of $\operatorname{CPD}(G) \cap L^1(G, \omega_G)$, hence it is the inverse Fourier transform of the function

$$\chi \mapsto \hat{\mu}(\chi) \Big(\int_V \overline{\chi} \, \mathrm{d}\omega_G \Big)^2$$

and consequently

$$\int \hat{\mu}(\chi) \Big(\int_V \overline{\chi} \, \mathrm{d}\omega_G \Big)^2 \omega_{G^\wedge}(\mathrm{d}\chi)$$
$$= (\mathbb{1}_V * \mathbb{1}_V * \mu)(0) = \int_G \omega_G(V \cap (V + x)) \, \mu(\mathrm{d}x) \leqslant \omega_G(V)\mu(2V).$$

As in the first part of this proof we look at this inequality with μ^n instead of μ, multiply on both sides by t^n for $t \in [0, 1[$ and sum over n. It follows that

$$\int_{G^\wedge} \Big(\int_V \overline{\chi} \, \mathrm{d}\omega_G \Big)^2 \frac{1}{1 - t\hat{\mu}(\chi)} \omega_{G^\wedge}(\mathrm{d}\chi) \leqslant \omega_G(V) \sum_{n \geqslant 0} t^n \, \mathsf{P}^0([X_n \in 2V]).$$

Now we choose a compact symmetric neighbourhood $W \in \mathfrak{V}_{G^\wedge}(0)$ and the neighbourhood V such that $\kappa(2V) < \infty$ and $\operatorname{Re}\chi(x) \geqslant \frac{1}{2}$ for all $(x, \chi) \in V \times W$. We obtain that

$$\omega_G(V) \int_W \operatorname{Re} \frac{1}{1 - t\hat{\mu}} \, \mathrm{d}\omega_{G^\wedge} \leqslant 4 \sum_{n \geqslant 0} t^n \, \mathsf{P}^e([X_n \in 2V])$$

hence that

$$\limsup_{t \uparrow 1} \int_W \operatorname{Re} \frac{1}{1 - t\hat{\mu}} \, \mathrm{d}\omega_{G^\wedge} < \infty$$

which implies the assertion.   □

Corollary 6.3.11 *If $X(\mu)$ is transient, then for every compact neighbourhood $W \in \mathfrak{V}_{G^\wedge}(0)$ the function*

$$\operatorname{Re} \frac{1}{1 - \hat{\mu}}$$

is integrable over W.

Proof. For all $t \in [0, 1[$ and $\chi \in G^\wedge$ we have that

$$\mathrm{Re} \, \frac{1}{1 - t\hat{\mu}} \geqslant 0$$

and that

$$\lim_{t\uparrow 1} \mathrm{Re} \, \frac{1}{1 - t\hat{\mu}} = \mathrm{Re} \, \frac{1}{1 - \hat{\mu}} \, .$$

By the Fatou Lemma together with the theorem we obtain

$$\int_W \mathrm{Re} \, \frac{1}{1 - \hat{\mu}} \, \mathrm{d}\omega_{G^\wedge} \leqslant \liminf_{t\uparrow 1} \int_W \mathrm{Re} \, \frac{1}{1 - t\hat{\mu}} \, \mathrm{d}\omega_{G^\wedge} < \infty$$

at first for the neighbourhood W constructed in the proof of the theorem, but since

$$\mathrm{Re} \, \frac{1}{1 - \hat{\mu}}$$

is continuous and $\neq 0$ on $G^\wedge \setminus \{0\}$, also for every compact neighbourhood $W \in \mathfrak{V}_{G^\wedge}(0)$. $\qquad\square$

Remark 6.3.12 The Corollary implies that every compact group G is recurrent (compare Example 6.3.9.1).

In fact, if G is compact, then $G^\wedge$ is discrete, so that $\omega_{G^\wedge}(\{0\}) > 0$. But this contradicts the $\omega_{G^\wedge}$-integrability of $\mathrm{Re} \, \frac{1}{1-\hat{\mu}}$ over any compact neighbourhood $\in \mathfrak{V}_{G^\wedge}(0)$.

Applying the characterization theorem 6.3.10 we can now decide on recurrence or transience of random walks on a significant class of locally compact Abelian groups.

Theorem 6.3.13 *Let* $X(\mu)$ *be a random walk with law* $\mu \in M^1(G)$, *where* G *has the form*

$$G = \mathbb{R}^d \times \mathbb{Z}^e$$

for $d, e \geqslant 0$. *Then*

(i) *if* $d + e = 1$, *i.e.* $G = \mathbb{R}$ *or* $G = \mathbb{Z}$, *and if*

$$\int |x| \, \mu(\mathrm{d}x) < \infty$$

as well as

$$\int x \, \mu(\mathrm{d}x) = 0 \, ,$$

then $X(\mu)$ *is recurrent.*

(ii) *If* $d + e = 2$ *and*

$$\int |x|^2 \, \mu(\mathrm{d}x) < \infty$$

  as well as

$$\int x \, \mu(\mathrm{d}x) = 0 \, ,$$

  then $X(\mu)$ *is recurrent.*

(iii) *If* $d + e \geqslant 3$, *then* $X(\mu)$ *is transient.*

Proof. (i). One easily verifies the estimates

$$\operatorname{Re} \frac{1}{1 - t\hat{\mu}} \geqslant \frac{1 - t}{(\operatorname{Re}(1 - t\hat{\mu}))^2 + t^2 (\operatorname{Im} \hat{\mu})^2}$$

and

$$\begin{aligned}
(\operatorname{Re}(1 - t\hat{\mu}))^2 &= ((1 - t) + \operatorname{Re}(t(1 - \hat{\mu})))^2 \\
&\leqslant 2(1 - t)^2 + 2t^2 (\operatorname{Re}(1 - \hat{\mu}))^2
\end{aligned}$$

valid for all $t \in [0, 1[$. By the assumption on the first moment of μ we see that $\hat{\mu}$ is continuously differentiable and that $\hat{\mu}'(0) = 0$. Now we apply the Taylor expansion and obtain the existence of an $\alpha > 0$ such that for $y \in \mathbb{R}^{\wedge} \cong \mathbb{R}$ or $y \in \mathbb{Z}^{\wedge} \cong \mathbb{T} \cong \,] - \pi, \pi] \subset \mathbb{R}$ with $|y| < \alpha$ we have

$$|\operatorname{Im} \hat{\mu}(y)| \leqslant \varepsilon |y|$$

as well as

$$\operatorname{Re}(1 - \hat{\mu}(y)) \leqslant \varepsilon |y| \, .$$

It follows that

$$\begin{aligned}
\int_{-\alpha}^{\alpha} \operatorname{Re} \frac{1}{1 - t\hat{\mu}(y)} \, \lambda(\mathrm{d}y) &\geqslant (1 - t) \int_{-\alpha}^{\alpha} \frac{1}{2(1 - t)^2 + 3t^2 \varepsilon^2 y^2} \, \lambda(\mathrm{d}y) \\
&\geqslant \frac{1}{3} \int_{-\frac{\alpha}{1-t}}^{\frac{\alpha}{1-t}} \frac{1}{1 + \varepsilon^2 y^2} \, \lambda(\mathrm{d}y) \, ,
\end{aligned}$$

hence that

$$\lim_{t \uparrow 1} \int_{-\alpha}^{\alpha} \operatorname{Re} \frac{1}{1 - t\hat{\mu}(y)} \, \lambda(\mathrm{d}y) \geqslant \frac{\pi}{3\varepsilon} \, .$$

Now suppose that $X(\mu)$ is transient. Then there exists an $\alpha_0 > 0$ such that

$$\operatorname{Re} \frac{1}{1 - t\hat{\mu}(y)} > 0$$

for all $y \in [-\alpha_0, \alpha_0]$ and

$$\lim_{t \uparrow 1} \int_{-\alpha_0}^{\alpha_0} \operatorname{Re} \frac{1}{1 - t\hat{\mu}(y)} \lambda(\mathrm{d}y) =: M < \infty$$

For $\alpha < \alpha_0$, however,

$$\lim_{t \uparrow 1} \int_{-\alpha}^{\alpha} \operatorname{Re} \frac{1}{1 - t\hat{\mu}(y)} \lambda(\mathrm{d}y) \leqslant M < \infty,$$

but this contradicts the fact that ε was chosen arbitrarily.

(ii) is proved similarly to (i) by embedding $G^{\wedge} = \mathbb{R}^d \times \mathbb{T}^e$ into $\mathbb{R}^{d+e}$ with $d + e = 2$.

For the proof of

(iii) we embed $G^{\wedge} = \mathbb{R}^d \times \mathbb{T}^e$ into $\mathbb{R}^{d+e}$ with $d + e \geqslant 3$ and infer from part 1. of the proof of the transience criterion 6.2.1 that there exist a relatively compact neighbourhood $U \in \mathfrak{V}_{G^{\wedge}}(0)$ and a constant $c > 0$ satisfying $\operatorname{Re} \hat{\mu}(y) \geqslant 0$ and

$$\operatorname{Re}(1 - \hat{\mu}(y)) \geqslant c \|y\|^2$$

for all $y \in U$ But then for $t < 1$ we obtain

$$\int_U \operatorname{Re} \frac{1}{1 - t\hat{\mu}(y)} \lambda^{d+e}(\mathrm{d}y) \leqslant \int_U \operatorname{Re} \frac{1}{1 - \hat{\mu}(y)} \lambda^{d+e}(\mathrm{d}y)$$

$$\leqslant \frac{1}{c} \int_U \frac{1}{\|y\|^2} \lambda^{d+e}(\mathrm{d}y) < \infty,$$

provided $d + e \geqslant 3$ which implies that all random walks in $\mathbb{R}^d \times \mathbb{Z}^e$ with $d + e \geqslant 3$ are transient. $\qquad\square$

Examples 6.3.14 of groups in $\mathcal{R}$

6.3.14.1 All compact groups belong to $\mathcal{R}$.

6.3.14.2 All groups of the form $G = \mathbb{R}^d \times \mathbb{Z}^e$ with $d + e \leqslant 2$ are elements of $\mathcal{R}$.

Properties 6.3.15 of the class $\mathcal{R}$

6.3.15.1 *Any open subgroup of a group in $\mathcal{R}$ belongs to $\mathcal{R}$.*

In particular,

6.3.15.2 *every subgroup of a discrete group in $\mathcal{R}$ belongs to $\mathcal{R}$.*

For the proof of Property 6.3.15.1 we start with an adapted random walk $X(\mu)$ in G and consider the Markov chain $(Z_n)_{n \in \mathbb{N}}$ in an open subgroup H of G given by

$$Z_n := X_{R_H^n}$$

where the return times R_H^n into H are defined recursively by

$$R_H^n = R_H^{n-1} + R_H \circ \theta_{R_H^{n-1}} \quad (n \geqslant 2)$$

and

$$R_H^1 := R_H .$$

Since $X(\mu)$ is assumed to be recurrent,

$$\mathsf{P}^0([R_H^n < \infty]) = 1$$

for every $n \in \mathbb{N}$, $(Z_n)_{n \in \mathbb{N}}$ is a well-defined random walk $X(\nu)$ with law

$$\nu := (\mathsf{P}^0)_{R_H} \in M^1(H) .$$

Moreover, we have

$$\limsup_{n \to \infty}[Z_n \in O] = \limsup_{n \to \infty}[X_n \in O]$$

for all open subsets O of H. Therefore, the recurrence of $X(\mu)$ yields

$$\mathsf{P}^0(\limsup_{n \to \infty}[Z_n \in O]) = 1$$

which says that $X(\nu)$ is recurrent in H.

6.3.15.3 *Let H be a compact subgroup of G. Then $G \in \mathcal{R}$ if and only if $G/H \in \mathcal{R}$.*

In order to see this we take $\mu \in M^1(G)$ and let $\dot{\mu}$ denote the image of μ under the canonical homomorphism $\pi : G \to G/H$. Now, for each compact neighbourhood $V \in \mathfrak{V}_G(0)$ the sets VH and $\pi(VH)$ are compact neighbourhoods in $\mathfrak{V}_G(0)$ and in $\mathfrak{V}_{G/H}(\dot{0})$ respectively. The identity

$$\sum_{n \to 0} \mu^n(VH) = \sum_{n \geqslant 0} \dot{\mu}^n(\pi(VH))$$

yields the assertion.

Theorem 6.3.16 (Characterization of the class $\mathcal{R}$)
Let G be a second countable locally compact Abelian group which by the structure theorem 4.2.20 is of the form

$$G = \mathbb{R}^d \times G_1 ,$$

where G_1 contains a compact open subgroup K such that

$$G/\{0\} \times K \cong \mathbb{R}^d \times G_2$$

with a countable group G_2 of rank r.
    Then $G \in \mathcal{R}$ if and only if $d + r \leqslant 2$.

The proof of the theorem relies on

Theorem 6.3.17 (Dudley)
A countable (Abelian) group G belongs to $\mathcal{R}$ if and only if $\operatorname{rank}(G) \leqslant 2$.

Proof. 1. If $\operatorname{rank}(G) > 2$, then there exists a subgroup of G that is isomorphic to the group $\mathbb{Z}^3$ which by Theorem 6.3.13 (iii) is not recurrent. On the other hand we infer from Property 6.3.15.2 that $G \notin \mathcal{R}$.

2. Conversely, we assume that $\operatorname{rank}(G) \leqslant 2$. There exists a sequence $(a_n)_{n \in \mathbb{N}}$ generating G and such that a_{m+1} does not belong to the subgroup G_m generated by $\{a_1, \ldots, a_m\}$ $(m \geqslant 1)$. From the structure of finitely generated Abelian groups proved in the book [36] by Hewitt and Ross we infer that for every $m \geqslant 1$ the group G_m is of the form $\mathbb{Z}^2 \times F$ or $\mathbb{Z} \times F$ or F, where F denotes a finite group. Applying Theorem 6.3.13 and Property 6.3.15.3 this implies that all random walks to be defined in G_m are recurrent $(m \geqslant 1)$.

Now we define a sequence $(\mu^{(m)})_{m \geqslant 1}$ of measures $\mu^{(m)} \in M^1(G_m)$ by

$$\mu^{(1)}(a_1) = \mu^{(1)}(-a_1) = \frac{1}{2},$$
$$\mu^{(m)}(x) = (1 - q_m)\mu^{(m-1)}(x)$$

for $m \geqslant 2$ and all $x \in G_{m-1}$, and

$$\mu^{(m)}(a_m) = \mu^{(m)}(-a_m) = \frac{1}{2}q_m$$

(with a proper choice of q_m). To every $x \in G$ there exists an $m \geqslant 1$ such that $x \in G_m$. The number

$$\mu(x) := \prod_{i \geqslant m+1} (1 - q_i)\mu^{(m)}(x)$$

is independent of m. If the product $\prod_{i \geqslant 2} (1 - q_i)$ converges, then

$$\sum_{x \in G} \mu(g) = \lim_{m \to \infty} \sum_{x \in G_m} \mu(g)$$
$$= \lim_{m \to \infty} \prod_{i \geqslant m+1} (1 - q_i) = 1,$$

hence $\mu \in M^1(G)$. Plainly μ is adapted. Our aim will be to show that the numbers q_i can be chosen such that the random walk $X(\mu)$ with law μ is recurrent.

For every $m \geqslant 1$ let $X^{(m)}$ be the random walk with law μ_m defined by the canonical probability measure $\mathsf{P}^\nu_{(m)}$. We choose once and for all a sequence $(r_n)_{n \in \mathbb{N}}$ in $]0,1[$ such that

$$\prod_{n \geqslant 1} (1 - r_n) > 0$$

Next we choose a number $l_1 \in \mathbb{N}$ and a sequence $(\alpha_j^{(1)})_{j \in \mathbb{N}}$ in $]0,1[$ such that

$$\prod_{j \geqslant 1} (1 - \alpha_j^{(1)})^{l_1} \sum_{k=1}^{l_1} \mathsf{P}^0_{(1)}([X_k^{(1)} = 0]) > 1 \,.$$

For $q_2 := r_1 \wedge \alpha_1^{(1)}$ the random walk $X^{(2)}$ in G_2 leads to

$$\prod_{j \geqslant 1} (1 - \alpha_j^{(2)})^{l_2} \sum_{k=l_1+1}^{l_2} \mathsf{P}^0_{(2)}([X_k^{(2)} = 0]) > 1$$

with the proper choice of l_2 and $(\alpha_j^{(2)})_{j \in \mathbb{N}}$. With $q_3 := r_2 \wedge \alpha_2^{(1)} \wedge \alpha_2^{(2)}$ we obtain the random walk $X^{(3)}$ in G_3 satisfying an analogous inequality. The inductive process continues and ends up with the inequality

$$\prod_{j \geqslant 1} (1 - \alpha_j^{(n)})^{l_n} \sum_{k=l_{n-1}+1}^{l_n} \mathsf{P}^0_{(n-1)}([X_k^{(n-1)} = 0]) > 1 \qquad (2)$$

together with the definition

$$q_n := r_{n-1} \wedge \alpha_{n-1}^{(1)} \wedge \alpha_{n-1}^{(2)} \wedge \cdots \wedge \alpha_{n-1}^{(n-1)} \,.$$

The choices taken are justified by the recurrence of the random walks $X^{(n)}$.

It is clear that

$$\prod_{i \geqslant 1} (1 - q_i) > 0 \,.$$

For the potential kernel κ of $X(\mu)$ we have that

$$\kappa(e) \geqslant \sum_{k \geqslant 1} \mathsf{P}^0([X_k = 0])$$

$$= \sum_{n \geqslant 1} \sum_{k=l_{n-1}+1}^{l_n} \mathsf{P}^0([X_k = 0])$$

$$\geqslant \sum_{n \geqslant 1} \sum_{k=l_{n-1}+1}^{l_n} \prod_{i \geqslant n+1} (1 - q_i)^k \, \mathsf{P}^0_{(n-1)}([X_k^{(n-1)} = 0]) \,,$$

where the k-th entry in the finite sum can be interpreted as the probability for $X(\mu)$ to be in $0 \in G$ at time k without having left G_{n-1}.

Moreover, by (2) we obtain that

$$\kappa(e) \geqslant \sum_{n \geqslant 1} \sum_{k=l_{n-1}+1}^{l_n} \prod_{i \geqslant n+1} (l - q_i)^{l_n} \, \mathsf{P}^0_{(n-1)}([X_k^{(n)} = 0])$$

$$\geqslant \sum_{n \geqslant 1} \sum_{k=l_{n-1}+1}^{l_n} \prod_{i \geqslant 1} (1 - \alpha_i^{(n)})^{l_n} \, \mathsf{P}^0_{(n-1)}([X_k^{(n)} = 0]) = \infty \,,$$

thus $X(\mu)$ is recurrent, and $G \in \mathcal{R}$. $\qquad\qquad\qquad\qquad \square$

The **proof of Theorem 6.3.16** follows from Property 6.3.15.3 together with the equivalence that $\mathbb{R}^d \times G_2$ is recurrent if and only if $d + r \leqslant 2$, where for both implications Dudley's theorem 6.3.17 is applied.

Remark 6.3.18 If in the theorem 6.3.16 $d + r \leqslant 2$, then G contains the dense subgroup $\mathbb{Q}^d \times G_2$, which belongs to $\mathcal{R}$. Consequently we have further

Examples 6.3.19 of groups in $\mathcal{R}$.

6.3.19.1 $\mathbb{Q}^2 \times K$ with a compact group K belongs to $\mathcal{R}$.

6.3.19.2 The group $\mathbb{Q}_p$ of p-adic numbers belongs to $\mathcal{R}$.

6.4   Classification of transient random walks

In order to study the asymptotic behaviour of random walks with law on a locally compact Abelian group we need to modify their canonical construction by compactifying G in the sense of Alexandrov.

Considering a transition kernel P on $(G, \mathfrak{B}(G))$ and picking a point $\triangle$ not in G we extend P to $(G_\triangle, \mathfrak{B}(G_\triangle))$, where $G_\triangle := G \cup \{\triangle\}$ and $\mathfrak{B}(G_\triangle) := \sigma(\mathfrak{B}(G) \cup \{\triangle\})$, by

$$P(x, \{\triangle\}) := \begin{cases} 1 - P(x, G) & \text{if } x \neq \triangle \\ 1 & \text{if } x = \triangle. \end{cases}$$

In this situation the canonical (product) measurable space will be $(\Omega, \mathfrak{A})$ with

$$\Omega := G_\triangle^{\mathbb{Z}_+} \quad \text{and}$$

$$\mathfrak{A} := \mathfrak{B}(G_\triangle)^{\otimes \mathbb{Z}_+}.$$

Moreover, the sequence $(X_n)_{n \in \mathbb{Z}_+}$ of projections $X_n : G_\triangle^{\mathbb{Z}_+} \to G_\triangle$ will be supplemented by

$$X_\infty(\omega) := \triangle$$

for all $\omega \in \Omega$. So the measures P^ν for $\nu \in M^1(G)$ governing the Markov chain $(X_n)_{n \in \mathbb{Z}_+}$ with transition kernel P will be interpreted as measures on the (enlarged) measurable space $(\Omega, \mathfrak{A})$.

We start the discussion by proving *some general renewal results* for transient random walks $X(\mu)$ with law $\mu \in M^1(G)$ admitting a potential kernel K.

Proposition 6.4.1 *Let $\mathcal{F}_K'$ denote the set of accumulation points of*

$$\mathcal{F}_K := \{K(x, \cdot) : x \in G\}$$

obtained as $x \to \triangle$. Then

(i) $\mathcal{F}_K' \subset \{c\,\omega_G : c \geqslant 0\}$,
(ii) $\mathcal{F}_K' \ni 0$.

Proof.   (i).   From Corollary 6.3.8 we infer that $\mathcal{F}_K$ is τ_{v}-relatively compact. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in G with $x_n \to \triangle$ and

$$\tau_{\mathrm{v}} - \lim_{n \to \infty} K(x_n, \cdot) = \nu \in M_+(G).$$

Since

$$\tau_{\mathrm{v}} - \lim_{n\to\infty} \varepsilon_{x_n} = 0 \,,$$

the equality

$$\varepsilon_{x_n} * \kappa = \varepsilon_{x_n} + \varepsilon_{x_n} * \kappa * \mu$$

valid for all $n \in \mathbb{N}$, leads as $n \to \infty$ to

$$\nu = \nu * \mu \,.$$

Now, the Choquet–Deny theorem 6.1.12 applies (to the adapted random walk with law $\mu \in M^1(G)$), such that ν is μ-invariant and hence a Haar measure of the form $c\omega_G$ with $c \geqslant 0$.

(ii). Let $f \in C_+^{\mathrm{c}}(G)$ with $f \neq 0$. As a supermartingale $(Kf \circ X_n)_{n\geqslant 1}$ converges to 0 P-a.s., hence there exists an $\omega \in \Omega$ such that

$$\lim_{n\to\infty} Kf \circ X_n(\omega) = 0$$

and

$$\lim_{n\to\infty} X_n(\omega) = \Delta \,,$$

where $X_n(\omega) \neq \Delta$ for every $n \geqslant 1$. But by (i)

$$\tau_{\mathrm{v}} - \lim_{n\to\infty} K(X_n(\omega), \cdot) = \nu = c\omega_G$$

with $\nu(f) = 0$, hence $\nu = 0$. □

Proposition 6.4.2 *Let $f \in C_+^{\mathrm{c}}(G)$. Then*

(i) *Kf is uniformly continuous.*
(ii)

$$\lim_{x\to\triangle} (Kf(x+y) - Kf(x)) = 0$$

uniformly for all y from any compact subset C of G.

(iii)

$$\liminf_{x\to\triangle} Kf(x) = 0 \,.$$

(iv)

$$\lim_{x\to\triangle} Kf(x)\, Kf(-x) = 0 \,.$$

Since (i) *implies that the mapping $x \mapsto K(x, \cdot)$ is continuous, hence determining the set $\mathcal{F}'_K$ is equivalent to determining the closure $\overline{\mathcal{F}}_K$ of $\mathcal{F}_K$ in $M_+(G)$ (with respect to τ_{v}).*

Proof. (i). From the τ_{v}-relative compactness of $\mathcal{F}_K$ we infer that for every compact subset C of G there exists a constant $M_C > 0$ such that

$$K(x, C) \leqslant M_C$$

whenever $x \in G$. Let $H := \operatorname{supp}(f)$ and let V be a compact neighbourhood in $\mathfrak{V}_G(0)$. Since f is uniformly continuous, for any $\varepsilon > 0$ we can choose a compact $V_1 \in \mathfrak{V}(0)$, $V_1 \subset V$ such that

$$|f(x + y) - f(x)| < \varepsilon$$

for all $y \in V_1$ and all $x \in G$. For $y \in V_1$ the function

$$x \mapsto g(x) := f(x + y) - f(x)$$

vanishes outside the compact set $C := H + V$, and

$$Kg(x) = Kf(x + y) - Kf(x)$$

whenever $x \in G$. Now, from the boundedness of K we deduce that

$$|Kf(x + y) - Kf(x)| \leqslant \varepsilon M_C$$

for all $x \in G$, $y \in V$, and this implies the assertion.

(ii). Proposition 6.4.1 tells us that every sequence in G converging to $\triangle$ (as $n \to \infty$) contains a subsequence $(x_n)_{n \in \mathbb{N}}$ such that

$$\tau_{\mathrm{v}} - \lim_{n \to \infty} K(x_n, \cdot) = c \omega_G$$

for some $c \geqslant 0$. But then for $y \in G$

$$\lim_{n \to \infty} (Kf(x_n + y) - Kf(x_n)) = c \left(\int f \, \mathrm{d}(\omega_G * \varepsilon_y) - \int f \, \mathrm{d}\omega_G \right) = 0$$

which shows that this limit exists. For every $\varepsilon > 0$ there exists a compact subset C_y of G satisfying

$$|Kf(x + y) - Kf(x)| < \frac{\varepsilon}{2}$$

whenever $x \in \complement C_y$.

Since by (i) Kf is uniformly continuous, for each $\varepsilon > 0$, every $y \in G$ has a neighbourhood $U_y \in \mathfrak{V}_G(y)$ such that for $z \in U_y$ we have

$$|Kf(x + z) - Kf(x + y)| < \frac{\varepsilon}{2}$$

uniformly in $x \in G$. But then

$$|Kf(x + z) - Kf(x)| < \varepsilon$$

whenever $z \in U_y$ and $x \in \complement C_y$. Covering the given compact set C by finitely many of the sets U_y for $y \in G$ yields the assertion.

(iii). With the abbreviation

$$a := \inf_{x \in G} Kf(x)$$

we have that

$$a \leqslant P^n Kf(x)$$

for every $n \geqslant 1$ and all $x \in G$. Letting n tend to ∞ this yields $a = 0$.

Now let Kf be strictly positive. Then

$$\liminf_{x \to \triangle} Kf(x) = 0 \,.$$

If Kf is not strictly positive, then there exists $x_0 \in G$ such that $Kf(x_0) = 0$, hence that

$$P^n Kf(x_0) = 0$$

for every $n \in \mathbb{N}$. But for any compact subset C of G there exists an $n \in \mathbb{Z}_+$ such that

$$P^n \big(x_0, \complement C\big) > 0 \,,$$

hence there is an $x \in \complement C$ with $Kf(x) = 0$. Thus, also in this case we arrive at the assertion.

(iv). Fixing $\varepsilon > 0$ we obtain from (ii) that there is a compact subset C_ε of G such that for $x \in \complement C_\varepsilon$ and for every $y \in D := \mathrm{supp}(f)$ we have

$$Kf(x) \leqslant Kf(x + y) + \varepsilon \,,$$

hence

$$\frac{Kf(x)\,Kf(y)}{\|Kf\|} \leqslant Kf(x + y) + \varepsilon \tag{1}$$

By the maximum principle 6.3.6.1 we obtain the validity of (1) for all $x \in \complement C_\varepsilon$ and all $y \in G$.

Now (iii) provides us with a point $y_\varepsilon \in D$ such that

$$Kf(y_\varepsilon) \leqslant \varepsilon \,.$$

Replacing y by $-x + y_\varepsilon$ in (1) this yields

$$Kf(x)\,Kf(-x + y_\varepsilon) \leqslant 2\|Kf\|\varepsilon \,.$$

But

$$\lim_{x \to \triangle} Kf(-x) - Kf(-x + y_\varepsilon) = 0 \,,$$

thus (iv) has been established. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 6.4.3 *There exists at most one measure $\mu \neq 0$ in $\mathcal{F}'_K$.*

Proof.   We assume that there exist two non-zero accumulation points in $\mathcal{F}'_K$. Then there exists a function $f \in C^c_+(G)$ with $D := \operatorname{supp}(f)$ such that the set

$$\{Kf(x) : x \in G\}$$

admits two limit points > 0 as $x \to \triangle$, where the smaller one will be denoted by l. Fixing $\varepsilon > 0$ we now construct inductively a sequence $(x_n)_{n \in \mathbb{N}}$ in G such that the support of the function

$$x \mapsto g(x) := f(x) + f(x + x_1) + \cdots + f(x + x_n)$$

is contained in $\bigcup_{i=0}^{n}(-x_i + D)$.

In fact, the construction starts with $x_0 := 0$. Then, by Proposition 6.4.2 (ii) and (iv) we choose $x_1 \in G$ such that

$$Kf(x_1) < l + \frac{\varepsilon}{2^2} \,,$$

$$Kf(-x_1) < \frac{\varepsilon}{2^2} \,,$$

$$|Kf(x_1) - Kf(x_1 + x)| < \frac{\varepsilon}{2^2}$$

and

$$|Kf(-x_1) - Kf(-x_1 + x)| < \frac{\varepsilon}{2^2} \,,$$

the latter two inequalities hold true for all $x \in D$. Since the sets $D \cup (-x_1 + D)$ and $D \cup (x_1 + D)$ are compact we can choose $x_2 \in G$ satisfying analogous inequalities with x_1 replaced by x_2, 2^2 by 2^3 and D by $D \cup (-x_1 + D)$ and $D \cup (x_1 + D)$ respectively, and so on until we arrive at an $x_n \in G$ satisfying the inequalities

$$Kf(x_n) < l + \frac{\varepsilon}{2^{n+1}} \,,$$

$$Kf(-x_n) < \frac{\varepsilon}{2^{n+1}} \,,$$

$$|Kf(x_n) - Kf(x_n + x)| < \frac{\varepsilon}{2^{n+1}}$$

and

$$|Kf(-x_n) - Kf(-x_n + x)| < \frac{\varepsilon}{2^{n+1}} \,,$$

the latter two inequalities being valid for all $x \in \bigcup_{i=0}^{n-1}(-x_i + D)$ or $x \in \bigcup_{i=0}^{n-1}(x_i + D)$ respectively. Clearly,

$$\operatorname{supp}(g) \subset \bigcup_{i=0}^{n}(-x_i + D).$$

Now, let $x \in -x_p + D$ for some $1 \leqslant p \leqslant n$. Then

$$Kf(x + x_p) \leqslant \|Kf\|,$$
$$Kf(x + x_{p+i}) = Kf(x_{p+i}) + Kf(x + x_{p+i}) - Kf(x_{p+i})$$
$$\leqslant l + \frac{\varepsilon}{2^{(p+i)+1}} + \frac{\varepsilon}{2^{(p+i)+1}}$$

whenever $i = 1, \ldots, n - p$, and

$$Kf(x + x_{p-i}) = Kf(z - x_p),$$

whenever $z = x_p + x + x_{p-i} \in x_{p-i} + D$ and $i = 1, \ldots, p$, so that

$$Kf(-x_p) + Kf(z - x_p) - Kf(-x_p) \leqslant \frac{\varepsilon}{2^{p+1}} + \frac{\varepsilon}{2^{p+1}}.$$

Forming the sum over all these inequalities we obtain that

$$Kg(x) \leqslant \|Kf\| + nl + \varepsilon$$

for all $x \in \bigcup_{i=0}^{n}(-x_i + D)$. The maximum principle 6.3.6.1 implies that this inequality holds for all $x \in G$. Another application of Proposition 6.4.2 (ii) yields

$$\limsup_{x \to \triangle} Kg(x) = (n+1) \limsup_{x \to \triangle} Kf(x),$$

hence

$$\limsup_{x \to \triangle} Kf(x) \leqslant \frac{nl}{n+1} + \frac{\|Kf\| + \varepsilon}{n+1}$$

for every $n \in \mathbb{N}$. Since l was the smallest of the presupposed two accumulation points > 0, the desired contradiction has been achieved. $\square$

The previous theorem justifies the following

Definition 6.4.4 A transient random walk $X(\mu)$ in G with potential kernel K is said to be **of type I** if

$$\tau_v - \lim_{x \to \triangle} K(x, \cdot) = 0.$$

$X(\mu)$ is said to be **of type II** if $\mathcal{F}'_K$ contains a measure $c\omega_G \neq 0$ (for $c > 0$).

Remark 6.4.5 Theorem 6.4.3 implies the dichotomy that every random walk in G is either of type I or of type II.

Clearly, any *symmetric* random walk $X(\mu)$ in G in the sense that its determining measure $\mu \in M^1(G)$ is symmetric, is of type I.

In view of a general approach towards characterizing random walks of type I we first treat the special (classical) cases $G = \mathbb{R}$ and $G = \mathbb{Z}$. We shall apply the symbol λ for the Lebesgue measure on $\mathbb{R}$ as well as for the counting measure on $\mathbb{Z}$. The Alexandrov compactification $G_\triangle$ of G will be understood as $G \cup \{-\infty, \infty\}$.

Concerning the *renewal of the groups* $\mathbb{R}$ and $\mathbb{Z}$ we first prove

Proposition 6.4.6 *Let G be $\mathbb{R}$ or $\mathbb{Z}$, and let $X(\mu)$ be a transient random walk in G with potential kernel K. Then*

(i)
$$\tau_{\mathrm{v}} - \lim_{x \to \pm\infty} K(x, \cdot) = c_\pm \, \lambda \,.$$

(ii) *At least one of the constants c_+ and c_- equals 0 the other one being > 0 or $= 0$.*

Proof. 1. Let $G = \mathbb{Z}$, and suppose that there exist two measures in $\mathcal{F}'_K$ (as $x \to \infty$). By Theorem 6.4.3 these are the measures 0 and $c\omega_G$ for $c > 0$. But then for any function $f \in C^{\mathrm{c}}_+(G)$, any $\varepsilon > 0$ and for sufficiently large x we have either

$$Kf(x) > c\omega_G(f) - \varepsilon$$

or

$$Kf(x) < \varepsilon \,.$$

Consequently there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in G with $x_n \to \infty$ satisfying the above inequalities with x replaced by $x_n + 1$ and x_n respectively, for all $n \in \mathbb{N}$. Hence Proposition 6.4.2 (ii) implies a contradiction.

2. For the case $G = \mathbb{R}$ we note that if there are two measures 0 and $c\omega_G$ for $c > 0$ in $\mathcal{F}'_K$, then for any $f \in C^{\mathrm{c}}_+(G)$, $c\omega_G$ is contained in $\mathcal{F}_K$ whenever $c \in [0, \omega_G(f)]$, since Kf is continuous by Proposition 6.4.2 (i). The desired contradiction follows from Theorem 6.4.3. $\qquad\square$

A function f on $\mathbb{R}$ is called *directly Riemann integrable* (R-*integrable*) if the series $\underline{\sigma}(f, h)$ and $\overline{\sigma}(f, h)$ defined below converge and if for every $\varepsilon > 0$

$$\overline{\sigma}(f, h) - \underline{\sigma}(f, h) < \varepsilon$$

whenever $h > 0$ is sufficiently small. Here, for $h > 0$ we define $\underline{U}_n(f, h)$ and $\overline{U}_n(f, h)$ as the minimum and the maximum of f taken over $[(n-1)h, nh]$ $(n \in \mathbb{N})$ and set

$$\underline{\sigma}^{(-)}(f, h) := h \sum_{h=-\infty}^{\infty} \underline{U}_n^{(-)}(f, h)$$

respectively. Clearly, the set $R(\mathbb{R})$ of R-integrable functions on $\mathbb{R}$ contains $C^c(\mathbb{R})$. For functions f on $\mathbb{R}$ that vanish on $]-\infty, 0[$, decrease on $[0, \infty[$ and satisfy $f(\infty) = 0$ the series $\underline{\sigma}$ and $\overline{\sigma}$ either both diverge or both converge. Thus $f \in R(\mathbb{R})$ if and only if $f \in L^1(\mathbb{R}, \lambda)$.

Proposition 6.4.7 *If* $f \in R(\mathbb{R})$, *then* Kf *is bounded, and*
$$\lim_{x \to \pm\infty} Kf(x) = c_\pm \, \lambda(f) \,.$$

Proof. 1. We first show the assertion for functions of the form
$$f_n := \mathbb{1}_{[(n-1)h, nh[}$$
for fixed $h \in \mathbb{R}_+^\times$ $(n \in \mathbb{N})$. By the maximum principle 6.3.6.1 there exists a constant $M > 0$ such that $Kf_n \leqslant M$ for all $n \in \mathbb{N}$. Let $(a_n)_{n \in \mathbb{Z}}$ denote a sequence in $\mathbb{R}_+^\times$ such that $\sum_{n=-\infty}^\infty a_n < \infty$. Then for
$$f := \sum_{n=-\infty}^{\infty} a_n f_n$$
we obtain that
$$\sum_{k=-m}^{m} a_k Kf_k(x) \leqslant Kf(x) \leqslant \sum_{k=-m}^{m} a_k Kf_k(x) + M \sum_{|k|>m} a_k$$
for all $x \in G$ and $m \in \mathbb{N}$, which implies the boundedness of Kf. The asserted convergence follows from the τ_v-convergence of the potential kernels $K(x, \cdot)$ as $x \to \infty$ $(x \to -\infty)$.

2. Let now $f \in R(\mathbb{R})$, $f > 0$. Setting
$$\underline{f}^{(-)} = \sum_{n=-\infty}^{\infty} \underline{U}_n^{(-)} f_n$$
we immediately see that
$$K\underline{f} \leqslant Kf \leqslant K\overline{f} \,,$$
hence that
$$\lim_{x \to \pm\infty} K\underline{f}(x) \leqslant \liminf_{x \to \pm\infty} Kf(x) \leqslant \limsup_{x \to \pm\infty} Kf(x) \leqslant \lim_{x \to \pm\infty} K\overline{f}(x)$$
and finally that
$$c_\pm \, \underline{\sigma} \leqslant \liminf_{x \to \pm\infty} Kf(x) \leqslant \limsup_{x \to \pm\infty} Kf(x) \leqslant c_\pm \, \overline{\sigma}$$
for all $x \in G$. This implies both assertions of the Proposition at once. $\square$

It follows the *computation of the constants* c_+ and c_-.

Proposition 6.4.8 *Let* $X(\mu)$ *admit a first moment in the sense that*

$$\int |x|\,\mu(\mathrm{d}x) < \infty\,,$$

in which case the mean

$$m := \int x\,\mu(\mathrm{d}x)$$

(of $X(\mu)$*) exists. Then* $X(\mu)$ *is of type II.*
  Moreover,

(1) $m \neq 0$, *since* $X(\mu)$ *is transient.*
(2) *If* $m > 0$ *(*$m < 0$*), then* $c_{\underset{(+)}{-}} = \frac{1}{m}$ *and* $c_{\underset{(-)}{+}} = 0$.
(3) *If* $\mathrm{supp}(\mu) \subset \overline{\mathbb{R}}_+$ *(*$:= \mathbb{R}_+ \cup \{\infty\}$*), then* $c_- = \frac{1}{m}$ *(and, of course* $c_+ = 0$*).*

Proof.   (1) follows from Theorem 6.3.13.
  In order to show the main statement and (2) of the Proposition we need to introduce two auxiliary functions g and k on $\mathbb{R}$ and $\mathbb{R}^2$ by

$$g(x) := \begin{cases} \mu([-x,\infty[) & \text{if } x \leqslant 0 \\ -\mu(]-\infty,-x[) & \text{if } x > 0 \end{cases}$$

and

$$k(x,y) := \begin{cases} 1 & \text{if } x < 0,\ y \geqslant -x \\ -1 & \text{if } x \geqslant 0,\ y < -x \\ 0 & \text{otherwise} \end{cases}$$

respectively. An application of the Fubini theorem to the integral

$$\int_{\mathbb{R}^2} k(x,y)\,\lambda(\mathrm{d}x)\,\mu(\mathrm{d}y)$$

for which

$$\int_{\mathbb{R}^2} |k(x,y)|\,\lambda(\mathrm{d}x)\,\mu(\mathrm{d}y) = \int_{-\infty}^{\infty} |y|\,\mu(\mathrm{d}y) < \infty$$

holds, yields that

$$\int_{-\infty}^{\infty} g(x)\,\lambda(\mathrm{d}x) = m\,.$$

Consequently, $g \in \mathrm{R}(\mathbb{R})$ and so Kg is bounded by Proposition 6.4.7. For $h := \mathbb{1}_{]-\infty,0[}$ we have

$$Ph(x) = \mu(]-\infty,-x[)$$

whenever $x \in \mathbb{R}$, hence

$$(I - P)h = g.$$

Applying the Choquet–Deny theorem 6.1.12 to the measures $\mu := l \cdot \lambda \in M^{sb}(\mathbb{R})$ with

$$l := h - Kg$$

and $\sigma := \mu \in M^1(\mathbb{R})$ we obtain that l is constant $= a$ $[\lambda]$. Now sequences $(x_n)_{\in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ can be found which converge to $-\infty$ and ∞ respectively and which satisfy $l(x_n) = l(y_n) = a$ for every $n \in \mathbb{N}$. On the other hand we have

$$h(z_n) = Kg(z_n) + l(x_n),$$

where z_n denotes either x_n or y_n. In the limit for $n \to \infty$ we obtain from Proposition 6.4.7 that

$$1 = c_- \, m + a$$

and

$$0 = c_+ \, m + a$$

according to the choices $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ respectively. But $c_+ > 0$ is impossible, since this yields $c_- = 0$, $a = 1$, hence $m < 0$. We therefore have $c_+ = 0$, hence $a = 0$ and consequently $c_- = \frac{1}{m}$.

As to the proof of (3) we first observe that the statement is true for $m < \infty$ by what we just showed. In the case $m = \infty$ we note that g vanishes on $]0, \infty[$, Kg is the smallest positive solution of the equation

$$(I - P)h = g,$$

and therefore $Kg \leqslant h$ (or even $Kg = h$ $[\omega_G]$). Moreover, for every $n \in \mathbb{N}$ the function

$$g_n := g \mathbb{1}_{[-n,0]} \in R(\mathbb{R})$$

and

$$Kg_n \leqslant Kg = h.$$

In the limit for $x \to -\infty$ this inequality leads to

$$c_- \int_{-\infty}^{\infty} g_n(x) \, \lambda(\mathrm{d}x) \leqslant 1$$

for all $n \in \mathbb{N}$. But this is a contradiction unless $c_- = 0$. $\qquad \square$

Given a random walk $X(\mu)$ in G with law $\mu \in M^1(G)$ and transition kernel P one introduces for any stopping time τ for $X(\mu)$ the **stopped transition kernel** P_τ by

$$P_\tau(x, B) := \mathsf{P}^x([X_\tau \in B])$$

for all $x \in G$, $B \in \mathfrak{B}(G)$. In the special case that τ is the **(first) entry time** (or hitting time) H_B of $X(\mu)$ into the set $B \in \mathfrak{B}(G)$ defined by

$$H_B(\omega) := \begin{cases} \inf\{n \in \mathbb{Z}_+ : X_n(\omega) \in B\} & \text{if } \{n \in \mathbb{Z}_+ : X_n(\omega) \in B\} \neq \emptyset \\ \infty & \text{otherwise,} \end{cases}$$

the strong Markov property of $X(\mu)$ implies that

$$P_{H_B} K f = K f \tag{2}$$

valid for all measurable functions $f \geqslant 0$ on G with $f = 0$ on $\complement B$. In fact, for such f and $x \in G$ one has

$$P_{H_B} K f(x) = \mathsf{E}^x(K f \circ X_{H_B})$$

$$= \mathsf{E}^x \left(\sum_{n \geqslant 0} P_n f \circ X_{H_B} \right)$$

$$= \mathsf{E}^x \left(\mathsf{E}_{X_{H_B}} \left(\sum_{n \geqslant 0} f \circ X_n \right) \right)$$

$$= \mathsf{E}^x \left(\sum_{n \geqslant 0} f \circ X_{n+H_B} \right)$$

$$= \mathsf{E}^x \left(\sum_{n \geqslant H_B} f \circ X_n \right)$$

$$= \mathsf{E}^x \left(\sum_{n \geqslant 0} f \circ X_n \right) - \mathsf{E}^x \left(\sum_{n < H_B} f \circ X_n \right)$$

$$= \mathsf{E}^x \left(\sum_{n \geqslant 0} f \circ X_n \right)$$

$$= K f(x).$$

In the following we shall write H_+ instead of $H_{\overline{\mathbb{R}}_+}$.

Properties 6.4.9

6.4.9.1 *If*

$$m^+ = \int_0^\infty x\, \mu(\mathrm{d}x) = \infty \,,$$

then

$$\tau_{\mathrm{v}} - \lim_{x \to -\infty} P_{H_+}(x, \cdot) = 0 \,.$$

In fact, if $\operatorname{supp}(\mu) \subset \mathbb{R}_+^{\times}$, then by Proposition 6.4.8.3

$$\lim_{x \to -\infty} K f(x) = 0 \,,$$

and since $Kf \geqslant f$, this yields the assertion. Let us now suppose that μ is not carried by $\mathbb{R}_+^{\times}$. We define a sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times τ_n for $X(\mu)$ by induction as follows:

$$\tau_0 := 0 \,,$$

$$\tau_k := \begin{cases} \inf\{n \in \mathbb{Z}_+ : X_n > X_{\tau_{k-1}}\} & \text{if } \{n \in \mathbb{Z}_+ : X_n > X_{\tau_{k-1}}\} \neq \emptyset \\ \infty & \text{otherwise} \end{cases}$$

for $k \geqslant 1$. Clearly, the random variables $X_{\tau_k} - X_{\tau_{k-1}}$ $(k \geqslant 1)$ are independent and equally distributed with distribution

$$\mu' := P_{\tau_1}(0, \cdot) = P_{H_+}(0, \cdot) \,.$$

Thus the sequence $(X_{\tau_k})_{k \in \mathbb{N}}$ forms a random walk $X(\mu')$ in $\mathbb{R}$ with $\operatorname{supp}(\mu') \subset \mathbb{R}_+^{\times}$, and since $\mu' \geqslant \operatorname{Res}_{\mathbb{R}_+^{\times}} \mu$, we obtain that

$$\int_0^{\infty} x \, \mu'(\mathrm{d}x) = \infty \,.$$

Now the distributions of entry into $\mathbb{R}_+^{\times}$ with respect to $X(\mu)$ and $X(\mu')$ coincide. The first part of this proof applied to μ' instead of μ yields the assertion.

6.4.9.2 *If*

$$m^- = \int_{-\infty}^0 (-x) \, \mu(\mathrm{d}x) < \infty \,,$$

and if $m > 0$, *then*

$$\mathsf{P}^x([H_+ < \infty]) = 1$$

for all $x \in \mathbb{R}$.

For the proof we assume that $m^+ < \infty$; the case $m^+ = \infty$ can be treated by truncating $X(\mu)$. Since $m^+ < \infty$, the strong law of large numbers implies

$$\lim_{n \to \infty} \frac{1}{n} X_n = m \quad [\mathsf{P}^0] \,.$$

Thus

$$\lim_{n \to \infty} X_n = \infty \quad [\mathsf{P}^0] \,,$$

and the assertion follows.

Theorem 6.4.10 (Renewal under non-existence of first moments)
If the transient random walk $X(\mu)$ in $G = \mathbb{R}$ or $G = \mathbb{Z}$ admits no first moment, then it is of type I.

Proof.   The arguments will be carried out for $G = \mathbb{R}$.

We suppose that $c_- > 0$, hence $c_+ = 0$, and that $m^+ = \infty$. Then by (2)

$$Kf(x) = \int_0^\infty P_{H_+}(x, \mathrm{d}y)\, Kf(y)$$

holds for each function $f \in C_+^c(\mathbb{R})$ vanishing on $]-\infty, 0[$ and all $x \in \mathbb{R}$. From $c_+ = 0$ we infer that given $\varepsilon > 0$ there exists an $a > 0$ such that $Kf(y) < \varepsilon$ for all $y \in \mathbb{R}$ with $y > a$. But then

$$Kf(x) \leqslant \int_0^a P_{H_+}(x, \mathrm{d}y)\, Kf(y) + \varepsilon$$

for all $x \in \mathbb{R}$. Moreover,

$$\tau_{\mathrm{v}} - \lim_{x \to -\infty} P_{H_+}(x, \cdot) = 0\,, \tag{3}$$

thus $Kf(x) \leqslant 2\varepsilon$ for sufficiently small $x \in \mathbb{R}$. This, however, contradicts the assumption that $c_- > 0$. It follows that $m^+ < \infty$, hence that $m^- = \infty$. Replacing $X(\mu)$ by its *dual* in the sense that $\mu \in M^1(\mathbb{R})$ is replaced by $\mu^\sim$, the above assumption is equivalent to supposing that $c_- = 0$, $c_+ > 0$ and $m^- < \infty$, $m^+ = \infty$. Then by

$$\mathsf{P}^x([H_+ < \infty]) = 1$$

for every $x \in \mathbb{R}$, (2) and (3) we obtain

$$
\begin{aligned}
0 &= c_- \cdot \lambda(f) \\
&= \lim_{x \to -\infty} Kf(x) \\
&= \lim_{x \to -\infty} \int_{\mathbb{R}} P_{H_+}(x, \mathrm{d}y)\, Kf(y) \\
&= \lim_{y \to \infty} Kf(y) \\
&= c_+ \cdot \lambda(f)\,.
\end{aligned}
$$

This is the desired contradiction.   □

The previous theorem extends to random walks in groups of the form $G = \mathbb{R} \times K$ and $G = \mathbb{Z} \times K$, where K is a compact group. We shall prove the extension only for $G = \mathbb{R} \times K$, where ω_G is chosen to be $\lambda \otimes \omega_K$ with $\omega_K \in M^1(K)$. Let p denote the canonical projection from

G onto $\mathbb{R}$. Then we shall employ the convention that $x \to +\infty$ $(-\infty)$ in G provided $p(x) \to +\infty$ $(-\infty)$ in $\mathbb{R}$.

Theorem 6.4.11 *Let* $G \cong \mathbb{R} \times K$ *or* $G \cong \mathbb{Z} \times K$, *and let* $X(\mu)$ *be a random walk with law* $\mu \in M^1(G)$ *which is of type II.*
   Then

(i)
$$\int |p(x)| \, \mu(\mathrm{d}x) < \infty \, .$$

(ii) *If*
$$m := \int_G p(x) \, \mu(\mathrm{d}x) > 0 \, ,$$

   then
$$\tau_{\mathrm{v}} - \lim_{x \to \triangle} K(x, \cdot) = \begin{cases} 0 & \text{if } \triangle = \infty \\ \frac{1}{m}\lambda & \text{if } \triangle = -\infty. \end{cases}$$

   For $\lambda < 0$ *there are symmetric limit relations.*

Proof.   From Proposition 6.4.1 we infer that for every sequence in G there exists a subsequence $(x_n)_{n \in \mathbb{N}}$ with $x_n := (y_n, k_n) \in \mathbb{R} \times K$ such that
$$\tau_{\mathrm{v}} - \lim_{n \to \infty} K(x_n, \cdot) = c \, \omega_G$$
with $c \geqslant 0$. Let $f \in C_+^c(G)$ be constant on the K-cosets of G. Then (ii) of Proposition 6.4.2 leads to
$$\lim_{x_n \to \infty} Kf(x_n) = \lim_{y_n \to \infty} Kf(y_n, 0) \, .$$
Now we consider the random walk $X(\tilde{\mu})$ with law $\tilde{\mu} := p(\mu) \in M^1(\mathbb{R})$. With the suggestive notation for $\tilde{f}$ and $\tilde{K}$ we obtain
$$Kf(y_n, 0) = \tilde{K}\tilde{f}(y_n)$$
for all $n \in \mathbb{N}$, and by the discussion starting with Proposition 6.4.8 we have
$$\lim_{y_n \to \infty} \tilde{G}\tilde{f}(y_n) = c_+ \cdot \lambda(\tilde{f})$$
where $c_+ > 0$ if and only if
$$\int_{-\infty}^{+\infty} |y| \, \tilde{\mu}(\mathrm{d}y) = \int_G |p(x)| \, \mu(\mathrm{d}x) < \infty \quad \text{and} > 0 \, .$$
As
$$\lambda(\tilde{f}) = \omega_G(f) \, ,$$
the theorem has been proved.   □

The final step of our analysis will be to establish the *renewal of random walks for general locally compact Abelian groups.*

We note that the (locally compact Abelian) group G remains to be second countable. The function f appearing in the proofs is always taken from $C_+^c(G)$ and $\neq 0$.

Theorem 6.4.12 *Let G_1 be a compactly generated, non-compact open subgroup of G such that G/G_1 is infinite.*

Then all transient random walks in G are of type I.

Proof. We suppose that the given random walk in G with potential kernel K is of type II and at the same time that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in G such that

$$\tau_{\mathrm{v}} - \lim_{n \to \infty} K(x_n, \cdot) = c\,\omega_G$$

for $c > 0$. The aim is to derive a contradiction.

Given the sequence $(x_n)_{n \in \mathbb{N}}$ we may assume without loss of generality that the G_1-cosets $G_1 + x_n$ of G $(n \in \mathbb{N})$ are pairwise disjoint.

In fact, if there is no subsequence of $(x_n)_{n \in \mathbb{N}}$ with this property, then there exists a G_1-coset containing infinitely many x_n, hence $(x_n)_{n \in \mathbb{N}}$ is contained in one (and the same) G_1-coset. Now we choose a sequence $(y_k)_{k \in \mathbb{N}}$ in G such that the cosets $G_1 + y_k$ $(k \in \mathbb{N})$ are pairwise disjoint. From (ii) of Proposition 6.4.2 we infer that for every $k \in \mathbb{N}$ we have that

$$\lim_{n \to \infty} Kf(x_n + y_k) = c\,\omega_G(f)\,,$$

hence that

$$|Kf(x_{n_k} + y_k) - c\,\omega_G(f)| < \frac{1}{2^k}$$

for some $n_k \in \mathbb{N}$. The sequence $(z_k)_{k \in \mathbb{N}}$ with $z_k := x_{n_k} + y_k$ fulfills the requirements.

Since G_1 is compactly generated and non-compact, there exists an $x \in G_1$ such that $nx \to \triangle$ as $n \to \infty$. Applying properties (ii) and (iv) of Proposition 6.4.2 one obtains

$$\lim_{n \to \infty} Kf(nx) = 0$$

or

$$\lim_{n \to \infty} Kf(-nx) = 0\,.$$

Without loss of generality we restrict our subsequence arguments to the first limit relationship. For every n there is a smallest integer m_n such that

$$Kf(x_n + n'x) \leqslant \frac{1}{2} c\omega_G(f)$$

whenever $n' \geqslant m_n$. This follows again from (ii) of Proposition 6.4.2. But for fixed l

$$\lim_{n \to \infty} Kf(x_n + lx) = c\omega_G(f),$$

hence $m_n > 1$ for sufficiently large n. We obtain, for such n,

$$Kf(x_n + m_nx - x) > \frac{1}{2} c\omega_G(f)$$

and

$$Kf(x_n + m_nx) \leqslant \frac{1}{2} c\omega_G(f).$$

Since G_1 is open, hence G/G_1 is discrete and every subset of G is contained in the union of finitely many G_1-cosets of G,

$$\lim_{n \to \infty} (x_n + m_nx - x) = \Delta.$$

Consequently,

$$\lim_{n \to \infty} Kf(x_n + m_nx - x) = c\omega_G(f)$$

and

$$\lim_{n \to \infty} Kf(x_n + m_nx) = 0.$$

By Proposition 6.4.2 (ii) this is a contradiction. $\square$

Corollary 6.4.13 *If* $G \cong \mathbb{R}^d \times \mathbb{Z}^e \times K$ *with* $d + e > 1$ *and a compact group* K, *then all transient random walks on* G *are of type I.*

Proof. From the proof of Theorem 6.4.11 we infer that it suffices to consider groups of the form $G = \mathbb{R}^d \times \mathbb{Z}^e$ for $d + e > 1$. If $e \geqslant 1$, the above theorem yields the result. If, however, $G = \mathbb{R}^d$ for $d > 1$, then $G = \bigcup_{n \geqslant 1} G_n$ for an increasing sequence $(G_n)_{n \in \mathbb{N}}$ of compact subsets G_n of G such that $\complement G_n$ is connected $(n \in \mathbb{N})$. Suppose that for $f \in C_+^c(G)$, $f \neq 0$ we have

$$\lim_{x \to \triangle} Kf(x) = a \neq 0.$$

Then every point in $[0, a]$ is a limit point of $Kf(x)$ (for $x \to \triangle$). But this contradicts the statement of Theorem 6.4.3. $\square$

Theorem 6.4.14 *If every element of G is compact, then all transient random walks in G are of type I.*

Proof.   As in the proof of the previous theorem we assume that there is a random walk of type II in G and aim at deriving a contradiction. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in G such that

$$\tau_{\mathrm{v}} - \lim_{n \to \infty} K(x_n, \cdot) = c\,\omega_G$$

with $c > 0$.   For every $n \in \mathbb{N}$ the set $S := \{kx_n : k \in \mathbb{N}\}^-$ is a compact subsemigroup, hence a subgroup of G .   Indeed, if 0 is not an accumulation point of S , then S will be discrete. Since S is compact, this is a contradiction. But 0 being an accumulation point of S , $-x$ is also one. This shows that $S = -S$, hence that S is a group.

   As a consequence of this we may also choose a sequence $(k_p)_{p \in \mathbb{N}}$ in $\mathbb{N}$ such that

$$\lim_{p \to \infty} k_p\, x_n = -x_n\,.$$

Thus, for every $\varepsilon > 0$ there is $k_n \in \mathbb{N}$ satisfying

$$Kf(k_n x_n) \leqslant Kf(-x_n) + \frac{\varepsilon}{n}\,.$$

By (iv) of Proposition 6.4.2

$$\lim_{n \to \infty} Kf(-x_n) = 0\,,$$

hence

$$\lim_{n \to \infty} Kf(k_n x_n) = 0\,.$$

Now we choose a constant M such that $M > \|Kf\| \vee \frac{1}{4}\,c\omega_G(f)$.   For sufficiently large $n \geqslant 1$ and the largest positive integer $m_n < k_n$ the inequalities

$$Kf(m_n x_n) \geqslant \frac{c^2\,\omega_G(f)^2}{4M}$$

and

$$Kf((m_n + 1)x_n) < \frac{c^2\,\omega_G(f)^2}{4M}$$

imply that

$$Kf(m_n x_n)\, Kf(x_n) \leqslant M(\varepsilon + Kf((m_n + 1)x_n))\,.$$

Furthermore, for sufficiently large n we have that

$$Kf(x_n) \geqslant \frac{1}{2}\,c\omega_G(f)$$

which leads to

$$Kf(m_nx_n) \leqslant \frac{2M}{c\,\omega_G(f)}(Kf((m_n+1)x_n)+\varepsilon)$$

$$\leqslant \frac{1}{2}c\,\omega_G(f) + \frac{2M\varepsilon}{c\,\omega_G(f)}$$

$$\leqslant \frac{1}{2}c\,\omega_G(f) + \frac{1}{2}\varepsilon$$

and

$$\frac{c^3\,\omega_G(f)^3}{8M^2} - \varepsilon \leqslant Kf((m_n+1)x_n)\,.$$

Now, by an appropriate choice of ε we arrive at the inequalities

$$0 < \alpha \leqslant Kf(m_nx_n) \leqslant \beta < c\,\omega_G(f)$$

and

$$0 < \gamma \leqslant Kf((m_n+1)x_n) \leqslant \delta < c\,\omega_G(f)$$

with constants α, β, γ and δ. Since $x_n \to \triangle$ as $n \to \infty$, at least one of the sequences $(m_nx_n)_{n\in\mathbb{N}}$ and $((m_n+1)x_n)_{n\in\mathbb{N}}$ admits a subsequence $(y_n)_{n\in\mathbb{N}}$ with $y_n \to \triangle$. But the sequence $(Kf(y_n))_{n\in\mathbb{N}}$ has at most 0 and $c\,\omega_G(f)$ for $c > 0$ as accumulation points. This shows the desired contradiction. □

Theorem 6.4.15 (General renewal theorem)
Let G be a second countable locally compact Abelian group, and suppose that there exists a random walk of type II in G. Then

$$G \cong \mathbb{R} \times K \quad or \quad G \cong \mathbb{Z} \times K\,,$$

where K is a compact group, and the renewal results of Theorem 6.4.11 apply.

Proof. From Theorem 6.4.14 we infer that there exists a non-compact element $y \in G$. Since by Appendix A.4.5 G admits a compactly generated open subgroup G_1, the subgroup

$$G_2 := [G_1 \cup \{y\}]$$

is non-compact, compactly generated and open. Thus, by Theorem 6.4.12 the group G/G_2 must be finite, hence G itself compactly generated. But then, by the structure theorem 4.2.19

$$G \cong \mathbb{R}^d \times \mathbb{Z}^e \times K$$

with a compact group K. Corollary 6.4.13 implies the assertion. □

Chapter 7

Hypergroups in Probability Theory

Hypergroups are locally compact spaces with a group-like structure on which the bounded measures convolve in a similar way to that on a locally compact group.

Prominent examples of hypergroups are locally compact spaces on which groups operate such as orbit spaces, double coset spaces and spaces of conjugacy classes. In hypergroups Dirac measures convolve to measures with not necessarily singleton support, a fact which necessitates developing an extended harmonic analysis for which only a restricted duality theory is available. This restriction, however, has suprising implications to probability theory, i.e. to the dynamics of random walks and more general processes with independent increments in a hypergroup.

Although the idea of an abstract convolution goes back to J. Delsarte's generalized translations studied in connection with second order partial differential operators in the late 30s, a useful general approach has been proposed only around 1975 as an axiomatic setting due to C.F. Dunkl, R.I. Jewett and R. Spector.

The reader who aims at learning more about the analysis on hypergroups and applications to probability theory than the present survey can provide is invited to consult the monograph [10] published by the author together with W.R. Bloom in 1995.

7.1   Commutative hypergroups

We begin with the definition of a hypergroup and first constructions of examples arising from groups. Then we introduce the generalized translation and give some implications to basic facts of harmonic analysis. For the analysis part we restrict the discussion to commutative hypergroups and

their duality properties.

I Introduction to hypergroups

In order to present the axioms of a hypergroup some **topological preparations** are needed. As for general terminology we start with a locally compact (Hausdorff) space E and introduce the symbol $\mathcal{K}(E)$ for the family of *nonempty* compact subsets of E. For $A, B \subset E$ we consider the family

$$\mathcal{K}_A(B) := \{C \in \mathcal{K}(E) : C \cap A \neq \emptyset, \; C \subset B\}\,.$$

Then $\mathcal{K}(E)$ can be given the topology generated by the subbasis consisting of all families $\mathcal{K}_U(V)$ for which U and V are open subsets of E. This topology τ_{M} introduced by E. Michael in 1951 as the *finite topology* enjoys the following

Properties 7.1.1

7.1.1.1 *If E is compact, then $\mathcal{K}(E)$ is $(\tau_{\mathrm{M}}\text{-})$ compact.*

7.1.1.2 $\mathcal{K}(E)$ *is a locally compact space.*

7.1.1.3 *The mapping*

$$x \mapsto \{x\}$$

is a homeomorphism of E onto a closed subset of $\mathcal{K}(E)$.

7.1.1.4 *The family of nonempty finite subsets of E is dense in $\mathcal{K}(E)$.*

7.1.1.5 *If W is a compact subset of $\mathcal{K}(E)$, then the union of all sets in W is a compact subset of E.*

Definition 7.1.2 If (E, d) is a metric space, one can introduce a metric on $\mathcal{K}(E)$ as follows: For $A \in \mathcal{K}(E)$ and $r \in \mathbb{R}_+^\times$ we define

$$V_r(A) := \{y \in E : d(x, y) < r \;\text{ for some }\; x \in A\}$$

and for $A, B \in \mathcal{K}(E)$

$$\varrho(A, B) := \{r \in \mathbb{R}_+^\times : A \subset V_r(B), \; B \subset V_r(A)\}\,.$$

ϱ is called the **Hausdorff metric**, and the corresponding topology with basis consisting of the sets

$$N_r(A) := \{B \in \mathcal{K}(E) : d(A, B) < r\}$$

for $A \in \mathcal{K}(E)$, $r \in \mathbb{R}_+^\times$ is said to be the **Hausdorff topology** τ_{H} on $\mathcal{K}(E)$.

It is easily verified that τ_{H} and τ_{M} coincide.

Definition 7.1.3 A **hypergroup** is a locally compact space K together with a **convolution** $*$ viewed as an associative bilinear mapping

$$(\mu, \nu) \mapsto \mu * \nu$$

from $M^{\mathrm{b}}(K) \times M^{\mathrm{b}}(K)$ into $M^{\mathrm{b}}(K)$ satisfying the following axioms

(HG1) $*$ is continuous with respect to the topology τ_{w} in $M^{\mathrm{b}}(K)$.

(HG2) For $x, y \in K$ the convolution $\varepsilon_x * \varepsilon_y$ belongs to $M_{\mathrm{c}}^1(K) := M^1(K) \cap M^{\mathrm{c}}(K)$.

(HG3) The mapping

$$(x, y) \mapsto \operatorname{supp}(\varepsilon_x * \varepsilon_y)$$

from $K \times K$ into $\mathcal{K}(K)$ is τ_{M}-continuous.

(HG4) There exists a **unit element** $e \in K$ such that

$$\varepsilon_e * \varepsilon_x = \varepsilon_x * \varepsilon_e = \varepsilon_x$$

for all $x \in K$.

(HG5) There exists an **involution**

$$x \mapsto x^-$$

on K extending to

$$\mu \mapsto \mu^-$$

on $M^{\mathrm{b}}(K)$ such that

$$(\varepsilon_x * \varepsilon_y)^- = \varepsilon_{y^-} * \varepsilon_{x^-}$$

and

$$e \in \operatorname{supp}(\varepsilon_x * \varepsilon_y) \quad \Longleftrightarrow \quad x = y^-$$

whenever $x, y \in K$.

A hypergroup K is called **commutative** if its convolution is commutative in the sense that

$$\varepsilon_x * \varepsilon_y = \varepsilon_y * \varepsilon_x$$

for all $x, y \in K$, and **hermitian** if its involution is the identity.

Given a hypergroup $(K, *)$ the space $(M^{\mathrm{b}}(K), *)$ is an involutive Banach algebra.

Remark 7.1.4 Due to the τ_{w}-denseness of the set of finitely supported measures in $M^{\mathrm{b}}(K)$ the convolution $*$ is uniquely determined by its values on the Dirac measures, i.e.

$$\mu * \nu(f) = \int_K \int_K T^x f(y)\, \mu(\mathrm{d}x)\, \nu(\mathrm{d}y)$$

whenever $\mu, \nu \in M^{\mathrm{b}}(K)$ and $f \in C^{\mathrm{b}}(K)$. Here

$$T^x f(y) := \varepsilon_x * \varepsilon_y(f)$$

for all $f \in C^{\mathrm{b}}(K)$, $y \in K$ defines the (x-left) *translation operator* T^x on $C^{\mathrm{b}}(K)$. The operator T^x can be extended to the space $B(K)$ of Borel measurable functions on K and also to the space $M(K)$ of all not necessarily bounded measures on K. The abbreviation

$$f(x * y) := T^x f(y)$$

properly interpreted for $f \in B(K)$, $x, y \in K$ is in common use.

For $A, B \subset K$ one defines

$$A * B := \bigcup \{\operatorname{supp}(\varepsilon_x * \varepsilon_y) : x \in A,\ y \in B\}$$

and works with this *convolution of sets* similar to sums of sets in the group case.

Before we enter the basics of harmonic analysis of commutative hypergroups we present

Examples 7.1.5 of hypergroups arising from groups.

We first note that every locally compact group G is clearly a hypergroup whose convolution structure is derived from the natural translation, i.e. the pointwise operation in G.

In fact, for $x, y \in G$

$$\varepsilon_x * \varepsilon_y = \varepsilon_{x \cdot y}.$$

In contrast to the deterministic operation of a group the hypergroup operation is a stochastic one in the sense that it is executed as a probability measure.

We recall a few **notations from group theory**.

A *continuous action* of a topological group H (with neutral element e) on a locally compact space E is a continuous mapping

$$(x, s) \mapsto x^s$$

from $E \times H$ to E such that

$$x^e = x$$

and

$$(x^s)^t = x^{st}$$

whenever $x \in E$, $s, t \in H$.

The orbit of $x \in E$ under H is defined by

$$x^H := \{x^s : s \in H\},$$

the totality of orbits by E^H.

Let G be a group. A mapping $\alpha : G \to G$ is called *affine* if there exist $c \in G$ and $\phi \in \mathrm{Aut}(G)$ such that

$$\alpha = c\,\phi.$$

If G and H are topological groups, a *continuous affine action* of H on G is defined as a continuous action

$$(x, s) \mapsto x^s$$

of H on G such that the mappings

$$x \mapsto x^s$$

$(s \in H)$ are affine.

Now suppose that E is a locally compact space, H a compact group, and that the mapping

$$(x, s) \mapsto x^s$$

is a continuous action of H on E. Then E^H is a decomposition of E into compact subsets and a closed subset of $\mathcal{K}(E)$. On E^H the quotient topology and the relative topology coincide, and E^H becomes a locally compact space. The canonical projection

$$x \mapsto x^H$$

is a continuous open mapping from E onto E^H.

7.1.5.1 Theorem. *Let G be a locally compact group and H a compact group. Suppose that*

$$(x, s) \mapsto x^s$$

is a continuous affine action of H on G. With the convolution given by

$$\varepsilon_{x^H} * \varepsilon_{y^H} := \iint \varepsilon_{(x^s y^t)^H}\, \omega_H(\mathrm{d}s)\, \omega_H(\mathrm{d}t)$$

for $x, y \in G$ and the quotient topology G^H satisfies axioms (HG1) to (HG3) of *Definition 7.1.3*.

7.1.5.2 Corollary. *If*

$$(x, s) \mapsto x^s$$

is a continuous action of H on G such that each of the mappings

$$x \mapsto x^s$$

belongs to $\mathrm{Aut}(G)$, *then* G^H *is a hypergroup with identity* $e^H = \{e\}$ *and involution*

$$x \mapsto (x^H)^- := (x^{-1})^H .$$

The convolution of G^H takes the form

$$\varepsilon_{x^H} * \varepsilon_{y^H} = \int_H \varepsilon_{(x^s y)^H} \, \omega_H(\mathrm{d}s)$$

$$= \int_H \varepsilon_{(xy^t)^H} \, \omega_H(\mathrm{d}t) .$$

7.1.5.3 Special cases.

(1) Let G be a compact group and let

$$G^G := \{x^G : x \in G\}$$

denote the set of conjugacy classes

$$x^G := \{t^{-1}xt : t \in G\}$$

of G furnished with the quotient topology. Then G^G is a commutative hypergroup with identity $\{e\}$ and involution

$$x \mapsto (x^G)^- := (x^{-1})^G$$

the convolution being of the form

$$\varepsilon_{x^G} * \varepsilon_{y^G} = \int_G \varepsilon_{(t^{-1}xty)^G} \, \omega_G(\mathrm{d}t)$$

for all $x, y \in G$.

(2) Let

$$G//H := \{HxH : x \in G\}$$

denote the *double coset space* associated with the pair (G, H) furnished with the quotient topology. Then $G//H$ becomes a hypergroup with identity $H = HeH$ and involution

$$x \mapsto (HxH)^- := Hx^{-1}H .$$

Its convolution appears to be

$$\varepsilon_{HxH} * \varepsilon_{HxH} = \int_H \varepsilon_{HxtyH}\, \omega_H(\mathrm{d}t)$$

whenever $x, y \in G$.

$G /\!/ H$ is commutative if (G, H) is a *Gelfand pair* in the sense that the subalgebra of H-biinvariant functions of $L^1(G, \omega_G)$ is commutative.

Examples of double coset hypergroups

(2.1) Let $G := \mathbb{M}(d)$ be the motion group of $\mathbb{R}^d$ and $H := \mathrm{SO}(d)$ the special orthogonal group $(d \geqslant 1)$. Then $G/H \cong \mathbb{R}^d$ and $G /\!/ H \cong \mathbb{R}_+$. $\mathbb{R}_+$ carries the *Bessel–Kingman convolution* inherited from the group convolution on $\mathbb{M}(d)$. (See 7.2.10.1.)

(2.2) Let $G := \mathrm{SO}_0(d)$ be the Lorenz group of dimension $d+1$ and $H := \mathrm{SO}(d)$. Then $G/H \cong \mathbb{H}^d$ which denotes the hyperbolic space of dimension d, and $G /\!/ H \cong \mathbb{R}_+$ where the convolution on $\mathbb{R}_+$ gives rise to a *Jacobi hypergroup* structure on $\mathbb{R}_+$. (See 7.2.10.2.)

(2.3) For $G := \mathrm{SO}(d+1)$ and $H := \mathrm{SO}(d)$ $(d \geqslant 1)$ we obtain $G/H \cong S^d$, the d-dimensional sphere, and the *Gegenbauer hypergroup* $G /\!/ H \cong [-1, 1]$. (See [10], 3.3.2.)

(2.4) If $G := \mathrm{Aut}(\Gamma)$ for a graph Γ and $H := H_{t_0}$ the stabilizer of the knot $t_0 \in \Gamma$ under the action of G on Γ, then $G/H \cong \Gamma$, and $G /\!/ H \cong \mathbb{Z}_+$ provides a *Cartier hypergroup*. (See [10], 3.3.20.)

II Some analysis on hypergroups

For developing harmonic analysis on any kind of algebraic–topological structure a canonical invariant measure has to be established. In the context of a hypergroup K we introduce (left) *Haar measures* as nonzero measures in $M_+(K)$ which are T^x-invariant for each $x \in K$.

Theorem 7.1.6 *Haar measures exist on hypergroups which are either discrete or compact or commutative.*

They are unique within a positive constant and have full support.

In what follows we shall fix for a given hypergroup K just one Haar measure and denote it by ω_K.

Examples 7.1.7

7.1.7.1 On a discrete hypergroup K a Haar measure ω_K can be normalized to satisfy

$$\omega_K(\{e\}) = 1$$

in which case

$$\omega_K(\{x\}) = \frac{1}{\varepsilon_{x^-} * \varepsilon_x(\{e\})}$$

for each $x \in K$.

7.1.7.2 If K is a compact hypergroup, ω_K is necessarily bounded, it can be normalzed to satisfy

$$\omega_K(K) = 1\,,$$

and it is an idempotent in $M^1(K)$.

   Moreover, any hypergroup admitting a bounded Haar measure must be compact.

Remark 7.1.8 Given a hypergroup K with a fixed Haar measure ω_K one can carry out the analysis in part analogous to the group case. The Banach spaces $L^p(K, \omega_K)$ $(1 \leqslant p \leqslant \infty)$ of p-times ω_K-integrable function classes can be furnished with a convolution as long as it is defined by

$$f * g(x) := \int_K (T^x f) g \, d\omega_K$$

for $f, g \in L^p(K, \omega_K)$, $x \in K$. $L^1(K, \omega_K)$ is in fact a Banach subalgebra of $M^{\mathrm{b}}(K)$ and as the *hypergroup algebra* of K the essential tool for developing a duality theory of hypergroups.

From now on all hypergroups to be considered will be commutative.

Definition 7.1.9 A continuous function χ on K is called **multiplicative** if $\chi(e) = 1$ and

$$\chi(x * y) = \chi(x)\,\chi(y)$$

holds for all $x, y \in K$.

   If in addition

$$\chi(x^-) = \overline{\chi(x)}$$

for all $x \in K$, then χ is called a **semicharacter**.

   A bounded semicharacter is said to be a **character** of K.

   By K^* and $K^\wedge$ we denote the sets of multiplicative functions on K and characters of K respectively.

Definition 7.1.10 The set $K^\wedge$ of all characters of K can be given the compact open topology τ_{co} in order to become a locally compact space.

$K^\wedge$ is called the **dual** space of K. The indicator $\mathbb{1}_K$ serves as the **unit character** $\mathbb{1}$ of K.

In general $K^\wedge$ does not admit a convolution structure, hence cannot be a hypergroup. The hypergroup property for $K^\wedge$ would require that for $\chi, \varrho \in K^\wedge$ there exists a measure $\mu_{\chi,\varrho} \in M^1(K^\wedge)$ satisfying

$$\chi(x)\,\varrho(x) = \int_{K^\wedge} \sigma(x)\,\mu_{\chi,\varrho}(\mathrm{d}\sigma)$$

for all $x \in K$. Only in this situation a convolution in $M^{\mathrm{b}}(K)$ can be defined by setting

$$\varepsilon_\chi \widehat{*}\, \varepsilon_\varrho := \mu_{\chi,\varrho}$$

whenever $\chi, \varrho \in K^\wedge$, and it can be attempted to verify the axioms of a hypergroup with $\mathbb{1}$ as the unit and some involution.

Hypergroups K with the property that $K^\wedge$ is also a hypergroup are called **strong**. For strong hypergroups K the **double dual** space $K^{\wedge\wedge} := (K^\wedge)^\wedge$ can be formed, and

$$K \subset K^{\wedge\wedge}.$$

If $K \cong K^{\wedge\wedge}$, then K is called a **Pontryagin hypergroup** (with reference to Pontryagin's duality theorem which is valid for all locally compact Abelian groups).

Despite the lack of a dual within the category of commutative hypergroups a generalized Fourier transform can be introduced.

Definition 7.1.11 There is a bijection

$$\chi \mapsto \tau_\chi$$

between the dual $K^\wedge$ of K and the symmetric part $\Delta_{\mathrm{s}}(L^1(K, \omega_K))$ of the Gelfand space $\Delta(L^1(K, \omega_K))$ of $L^1(K, \omega_K)$ given by

$$\tau_\chi(f) := \int_K \overline{\chi} f \, \mathrm{d}\omega_K$$

for all $f \in L^1(K, \omega_K)$. The mapping

$$\chi \mapsto \hat{f}(x) := \tau_\chi(f)$$

on $K^\wedge$ serves as the **Fourier transform** of $f \in L^1(K, \omega_K)$. More generally one introduces the Fourier transform

$$\chi \mapsto \widehat{\mu}(\chi)$$

of $\mu \in M^{\mathrm{b}}(K)$ by

$$\widehat{\mu}(\chi) := \int_K \overline{\chi}\, \mathrm{d}\mu$$

for all $\chi \in K^{\wedge}$.

The **Fourier mapping**

$$\mu \mapsto \widehat{\mu}$$

is a norm-decreasing involutive algebra isomorphism $\mathcal{F}$ from $M^{\mathrm{b}}(K)$ into $C^{\mathrm{b}}(K)$.

By the isometric embedding of the hypergroup algebra $L^1(K, \omega_K)$ into $M^{\mathrm{b}}(K)$ similar properties remain valid for

$$\mathrm{Res}_{L^1(K,\omega_K)}\, \mathcal{F}\,.$$

In particular, $M^{\mathrm{b}}(K)$ and $L^1(K, \omega_K)$ are semisimple Banach algebras.

On $L^1(K, \omega_K)$ the Fourier mapping enjoys the *Riemann–Lebesgue property*, i.e.

$$\mathcal{F}L^1(K, \omega_K) \subset C^0(K^{\wedge})\,.$$

A categorial property of significance is the duality between compact and discrete hypergroups: if K is compact, $K^{\wedge}$ is discrete; if K is discrete, K is compact.

It is a surprising fact that, for commutative hypergroups without any additional assumption on their dual space a Plancherel-type theorem holds.

Theorem 7.1.12 (Plancherel–Levitan)
*There exists a **Plancherel measure** $\pi_K \in M_+(K^{\wedge})$ such that*

$$\int_K |f|^2\, \mathrm{d}\omega_K = \int_{K^{\wedge}} |\hat{f}|^2\, \mathrm{d}\pi_K$$

for all $f \in L^1(K, \omega_K) \cap L^2(K, \omega_K)$.

In particuler, the Fourier mapping $\mathcal{F}$ can be extended to the space $L^2(K, \omega_K)$.

Discussion 7.1.13 In general $\mathrm{supp}\, \pi_K$ is a proper subset of $K^{\wedge}$, and it is this behavior that leads to additional efforts in developing harmonic analysis on a commutative hypergroup. The structure of $\mathrm{supp}\, \pi_K$ is yet to be properly understood. Still $\mathrm{supp}\, \pi_K$ can be described in terms of the (left) *regular representation*

$$\mu \mapsto \lambda(\mu)$$

on $M^{\mathrm{b}}(K)$ defined by

$$\lambda(\mu)f := \mu * f$$

with

$$\mu * f(x) := \int_K f(y^- * x)\,\mu(\mathrm{d}y)$$

for all $f \in L^2(K, \omega_K)$, $x \in K$.

The regular representation is a faithful norm-decreasing involutive representation of $M^{\mathrm{b}}(K)$, and

$$S := \{\chi \in K^{\wedge} : |\widehat{\mu}(\chi)| \leqslant \|\lambda(\mu)\| \text{ for all } \mu \in M^{\mathrm{b}}(K)\}$$

is a nonempty closed subset of $K^{\wedge}$.

It follows from the construction of the Plancherel–Levitan measure that in fact

$$S = \operatorname{supp} \pi_K .$$

The following

Consequences 7.1.14 of the Plancherel–Levitan theorem justify the notation of a *regular dual*

$$K^{\sim} := S = \operatorname{supp} \pi_K$$

of K.

7.1.14.1 $\mathcal{F}$ *is injective on* $K^{\sim}$, *i.e. for* $\mu, \nu \in M^{\mathrm{b}}(K)$ *with*

$$\operatorname{Res}_{K^{\sim}} \widehat{\mu} = \operatorname{Res}_{K^{\sim}} \widehat{\nu}$$

we have $\mu = \nu$.

7.1.14.2 $K^{\sim}$ *separates the points of* K.

We stress again that generally

$$K^{\sim} \subsetneqq K^{\wedge}$$

and that $K^{\sim}$ doesn't even contain the unit character $\mathbb{1}$. Exceptions to this deviation from the group case are compact hypergroups for which $K^{\sim} = K$ holds (see Remark 7.2.6).

Moreover,

7.1.14.3 *If* $K^{\wedge}$ *admits a convolution and* $\mathbb{1} \in K^{\sim}$, *then* $K^{\sim} = K^{\wedge}$.

Consequently,

7.1.14.4 *If* $K^{\wedge}$ *is a hypergroup then*

$$\pi_K = \omega_{K^{\wedge}} .$$

Definition 7.1.15 Dual to the Fourier transform $\widehat{\mu}$ of $\mu \in M^{\mathrm{b}}(K)$ there is the **inverse Fourier transform** $\check{\sigma}$ of $\sigma \in M^{\mathrm{b}}(K^{\wedge})$ defined by

$$\check{\sigma}(x) := \int_{K^{\wedge}} \chi(x)\,\sigma(\mathrm{d}\chi)$$

for all $x \in K$, which can also be evaluated for measures $\sigma := k \cdot \pi_K \in M^{\mathrm{b}}(K^{\wedge})$ to serve as the Fourier transform of $k \in L^1(K^{\wedge}, \pi_K)$.

Properties 7.1.16 of the inverse Fourier transform.

7.1.16.1 *The corresponding Plancherel–Levitan equality reads*

$$\int_K |\check{k}|^2\,\mathrm{d}\omega_K = \int_{K^{\wedge}} |k|^2\,\mathrm{d}\pi_K$$

for every $k \in L^1(K^{\wedge}, \pi_K) \cap L^2(K^{\wedge}, \pi_K)$,

7.1.16.2 *the inverse Fourier mapping* $\sigma \mapsto \check{\sigma}$ *is injective on* $M^{\mathrm{b}}(K^{\wedge})$,

7.1.16.3 *for* $f \in L^1(K, \omega_K) \cap C(K)$ *with* $\hat{f} \in L^1(K^{\wedge}, \pi_K)$ *one has*

$$f = (\hat{f})^{\vee},$$

and

7.1.16.4 *for* $f \in L^1(K^{\wedge}, \pi_K) \cap C(K^{\wedge})$ *with* $\check{f} \in L^1(K, \omega_K)$

$$f = (\check{f})^{\wedge}$$

holds on $K^{\sim}$.

7.2 Decomposition of convolution semigroups of measures

In this section we shall apply harmonic analysis on a commutative hyper-group K to convolution semigroups of probability measures on K, a useful measure-theoretic tool for studying stochastic processes in K. For the reader's convenience we start by presenting two basic constructions of convolutions on the sets $\mathbb{Z}_+$ and $\mathbb{R}_+$ which will yield polynomial and Sturm–Liouville hypergroups respectively.

I Constructions of hypergroups

Construction 7.2.1 Let $(a_n)_{n\in\mathbb{Z}_+}$, $(b_n)_{n\in\mathbb{Z}_+}$ and $(c_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}_+^\times$, $\mathbb{R}_+$ and $\mathbb{R}_+^\times$ respectively satisfying the relations

$$a_0 + b_0 = 1$$

and

$$a_n + b_n + c_n = 1$$

for all $n \in \mathbb{N}$. We define a sequence $(Q_n)_{n\in\mathbb{Z}_+}$ of polynomials on $\mathbb{R}$ by

$$Q_0(x) = 1$$

$$Q_1(x) = \frac{1}{a_0}(x - b_0) \tag{1}$$

$$Q_1(x)\, Q_n(x) = a_n Q_{n+1}(x) + b_n Q_n(x) + c_n Q_{n-1}(x) \qquad (n \in \mathbb{N})$$

for all $x \in \mathbb{R}$. A theorem of Favard yields the existence of a measure $\pi \in M^1(\mathbb{R})$ with the property that

$$\int_{\mathbb{R}} Q_n(x)\, Q_m(x)\, \pi(\mathrm{d}x) = s_m \delta_{mn}$$

with $s_m > 0$ (see [14]). Hence the sequence $(Q_n)_{n\in\mathbb{Z}_+}$ is orthogonal with respect to π, but not orthonormal, although

$$Q_n(1) = 1$$

for all $n \in \mathbb{Z}_+$.

Properties 7.2.2 *For all* $n, m, k \in \mathbb{Z}_+$ *with* $|n - m| \leqslant k \leqslant n + m$ *we have the* **real linearization**

$$Q_n(x)\, Q_m(x) = \sum_{k=|n-m|}^{n+m} g(n, m, k)\, Q_k(x) \tag{2}$$

$(x \in \mathbb{R})$, *where* $g(n, m, k) \in \mathbb{R}$.

Moreover, $g(n, m, |n - m|) \neq 0$ *and* $g(n, m, n + m) \neq 0$. *The co-efficients* $g(n, m, k)$ *can be recursively described in terms of the initial sequences* $(a_n)_{n\in\mathbb{Z}_+}$, $(b_n)_{n\in\mathbb{Z}_+}$ *and* $(c_n)_{n\in\mathbb{N}}$.

Theorem 7.2.3 *Let* $(Q_n)_{n \in \mathbb{Z}_+}$ *be an orthogonal sequence of polynomials defined by the recurrence relation* (1). *Assume that one has a* **nonnegative linearization** *of the form* (2) *in the sense that*

$$g(n, m, k) \geqslant 0$$

for all $n, m, k \in \mathbb{Z}_+$ *with* $|n - m| \leqslant k \leqslant n + m$.

Then a convolution $*_{(Q_n)}$ *can be introduced in* $M^1(\mathbb{Z}_+)$ *by*

$$\varepsilon_n *_{(Q_n)} \varepsilon_m := \sum_{k=|n-m|}^{n+m} g(n, m, k)\, \varepsilon_k$$

for $n, m \in \mathbb{Z}_+$ *which yields a hypergroup* $\big(\mathbb{Z}_+, *_{(Q_n)}\big)$ *with unit element* 0 *and involution* $n \mapsto n^\sim := n$.

The hypergroup $\big(\mathbb{Z}_+, *_{(Q_n)}\big)$ is said to be the **polynomial hypergroup induced by the polynomial sequence** $(Q_n)_{n \in \mathbb{Z}_+}$.

Properties 7.2.4 of polynomial hypergroups.

7.2.4.1 *From the above cited Theorem of Favard follows that every commutative hypergroup* $(\mathbb{Z}_+, *)$ *with* 0 *as unit element and the identity as involution satisfying*

$$\{n - 1, n + 1\} \subset \mathrm{supp}(\varepsilon_1 * \varepsilon_n) \subset \{n - 1, n, n + 1\}$$

for all $n \in \mathbb{Z}_+$ *is in fact a polynomial hypergroup induced by some orthogonal polynomial sequence.*

7.2.4.2 *In accordance with Example 7.1.7.1 the* Haar measure *of a polynomial hypergroup* $(\mathbb{Z}_+, *)$ *takes the form*

$$\omega(\{n\}) = \frac{1}{\varepsilon_n * \varepsilon_n(\{0\})}$$
$$= \frac{1}{g(n, n, 0)} = \frac{1}{s_n}$$

whenever $n \in \mathbb{Z}_+$.

7.2.4.3 *Concerning the* dual *of* $(\mathbb{Z}_+, *)$ *we introduce for every* $x \in \mathbb{R}$ *the mapping*

$$\alpha_x : \mathbb{Z}_+ \to \mathbb{C}$$

defined by

$$\alpha_x(n) := Q_n(x)$$

for all $x \in \mathbb{R}$ *and observe that with the notation*

$$D := \{x \in \mathbb{R} : |Q_n(x)| \leqslant 1 \ \text{ for all } \ n \in \mathbb{Z}_+\}$$

we obtain

$$\{\alpha_x : x \in D\} = \mathbb{Z}_+^{\wedge} .$$

In fact, the mapping $x \mapsto \alpha_x$ *from* D *into* $\mathbb{Z}_+^{\wedge}$ *is a homeomorphism admitting* $\alpha \mapsto a_0 \alpha(1) + b_0$ *as its inverse, and* D *is a compact subset of the interval* $[1 - 2a_0, 1]$.

7.2.4.4 *For the polynomial hypergroup* $(\mathbb{Z}_+, *_{(Q_n)})$ *the* Plancherel–Levitan theorem *of 7.1.12 takes on the following form:*

$$\sum_{n=0}^{\infty} |f(n)|^2 \, \omega(\{n\}) = \int_D |\hat{f}(x)|^2 \, \pi(\mathrm{d}x)$$

for every $f \in L^1(\mathbb{Z}_+) := \ell^1(\omega),$ *where* π *is the orthogonalization measure of the polynomial sequence* $(Q_n)_{n \in \mathbb{Z}_+},$ *with*

$$\operatorname{supp} \pi = \{x \in D : |\hat{f}| \leqslant \|\lambda(f \cdot \omega)\| \ \text{ for every } \ f \in \ell^1(\omega)\} .$$

(See 7.1.13.)

Examples 7.2.5 of polynomial hypergroups are **Jacobi polynomial hypergroups** $(\mathbb{Z}_+, *_{(Q_n)})$ where

$$Q_n = Q_n^{(\alpha, \beta)}$$

for all $n \in \mathbb{Z}_+,$ $\alpha \geqslant \beta \geqslant -1,$ $\alpha + \beta + 1 \geqslant 0,$ are the Jacobi polynomials introduced recursively via (1) by

$$a_0 := \frac{2\alpha + 1}{\alpha + \beta + 2} ,$$

$$b_0 := \frac{\beta - \alpha}{\alpha + \beta + 2} ,$$

$$a_n := \frac{2(n + \alpha + \beta + 1)(n + \alpha + 1)(\alpha + \beta + 2)}{(2n + \alpha + \beta + 2)(2n + \alpha + \beta + 1)2(\alpha + 1)} ,$$

$$b_n := \frac{\alpha - \beta}{2(\alpha + 1)} \left(1 - \frac{(\alpha + \beta + 2)(\alpha + \beta)}{(2n + \alpha + \beta + 2)(2n + \alpha + \beta)} \right) ,$$

and

$$c_n := \frac{2n(n + \beta)(\alpha + \beta + 2)}{(2n + \alpha + \beta + 1)(2n + \alpha + \beta)2(\alpha + 1)} \qquad (n \in \mathbb{N}) .$$

The Jacobi polynomials $Q_n^{(\alpha,\beta)}$ are orthogonal with respect to the measure
$$\pi(\mathrm{d}x) := c_{\alpha,\beta}\, \mathbb{1}_{[-1,1]}\, (1-x)^\alpha\, (1+x)^\beta\, \mathrm{d}x$$
with
$$c_{\alpha,\beta} := \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\,\Gamma(\alpha+1)\,\Gamma(\beta+1)}\,,$$
where π serves as the Plancherel(–Levitan) measure corresponding to the Haar measure ω of the Jacobi hypergroup given by
$$\omega(\{0\}) = 1\,,$$
$$\omega(\{n\}) = \frac{(2n+\alpha+\beta+1)\,(\alpha+\beta+1)_n\,(\alpha+1)_n}{(\alpha+\beta+1)\,n!\,(\beta+1)_n}$$
whenever $n \in \mathbb{N}$.

From the asymptotic behaviour of the gamma function one deduces that
$$\omega(\{n\}) = O(n^{2\alpha+1})\,,$$
hence that
$$\operatorname{supp}\pi = D = [-1,1]\,.$$
Moreover, for all $\alpha \geqslant \alpha > -1$ and either $\beta \geqslant -\frac{1}{2}$ or $\alpha + \beta \geqslant 0$ we have the existence of a convolution $\varepsilon_x \widehat{*} \varepsilon_y \in M^1([-1,1])$ such that
$$Q_n^{(\alpha,\beta)}(x)\, Q_n^{(\alpha,\beta)}(y) = \int_{[-1,1]} Q_n^{(\alpha,\beta)}\, \mathrm{d}(\varepsilon_x \widehat{*} \varepsilon_y)$$
for all $x, y \in [-1,1]$. Endowed with this convolution $[-1,1]$ becomes a compact hypergroup. Since $[-1,1]$ has been identified with $D = \mathbb{Z}_+^\wedge$, the resulting **dual Jacobi polynomial hypergroup** $(\mathbb{Z}_+^\wedge, \widehat{*})$ is in fact a Pontryagin hypergroup.

7.2.5.1 Subexamples of Jacobi polynomial hypergroups are the **ultraspherical hypergroups** which appear for the parameters $\alpha = \beta$.

Further specialization yields the **Chebyshev hypergroups of the first kind** for $\alpha = -\frac{1}{2}$, **of the second kind** for $\alpha = \frac{1}{2}$, and the **Legendre hypergroups** for $\alpha = 0$.

For integer parameters
$$\alpha = \beta = \frac{d-3}{2} \qquad \text{for } d \geqslant 2$$
one obtains the identifications
$$\mathbb{Z}_+ \cong (\mathrm{SO}(d)//\mathrm{SO}(d-1))^\wedge$$
and
$$\mathbb{Z}_+^\wedge \cong \mathrm{SO}(d)//\mathrm{SO}(d-1)\,.$$
(See 7.1.5.3 (2.3).)

Remark 7.2.6 There are polynomial hypergroups $(\mathbb{Z}_+, *)$ which are not strong. In particular for the Cartier hypergroups of 7.1.5.3 (2.4) and for the Karlin–McGregor hypergroups

$$\operatorname{supp} \pi \neq D \cong \mathbb{Z}_+^{\wedge} .$$

(See [91] and [52] respectively.)

Construction 7.2.7 A function $A \in C(\mathbb{R}_+) \cap C^1(\mathbb{R}_+^{\times})$ is said to be **admissible** if

$$A(x) > 0 \quad \text{for all} \quad x \in \mathbb{R}_+^{\wedge}$$

and if there are constants $\varepsilon > 0$, $\alpha_0 \geqslant 0$ and a function $\alpha_1 \in C^{\infty}(\,]{-}\varepsilon, \varepsilon[\,)$ satisfying

$$\frac{A'}{A}(x) = \frac{\alpha_0}{x} + x\,\alpha_1(x)$$

whenever $x \in \,]0, \varepsilon[$. In the singular case $\alpha_0 > 0$ we assume in addition that α_1 is even.

There is a **Sturm–Liouville operator** associated with an admissible function A defined by

$$Lf(x) := -\frac{1}{A(x)}\big(A(x)f'(x)\big)'$$

for all $f \in C^2(\mathbb{R}_+^{\times})$, $x > 0$.

Moreover we introduce the differential operator ℓ_A on $C^2\big((\mathbb{R}_+^{\times})^2\big)$ by

$$\ell_A(u)(x, y) := L_A^1 u(x, y) - L_A^2 u(x, y)$$

for all $u \in C^2\big((\mathbb{R}_+^{\times})^2\big)$, $x, y > 0$, where L_A^j denotes the Sturm–Liouville operation taken with respect to the j-th variable of the function at evaluation.

Definition 7.2.8 A hypergroup $(\mathbb{R}_+, *)$ is called a **Sturm–Liouville hypergroup** if there exists an admissible function A such that for each even function $f \in C^{\infty}(\mathbb{R})$ the function

$$(x, y) \mapsto u_f(x, y) := \int_{\mathbb{R}_+} f \, \mathrm{d}(\varepsilon_x * \varepsilon_y)$$

on $\mathbb{R}_+^2$ satisfies $u_f \in C^2\big((\mathbb{R}_+^{\times})^2\big)$ with

$$\ell_A(u_f) = 0$$

and

$$(u_f)_y(x, 0) = 0$$

whenever $x \in \mathbb{R}_+$.

Properties 7.2.9 of Sturm–Liouville hypergroups.

7.2.9.1 *For each hypergroup* $(\mathbb{R}_+, *)$ *the boundary point* 0 *is the unit element and the hypergroup involution is the identity. Consequently, each hypergroup* $(\mathbb{R}_+, *)$ *is commutative.*

7.2.9.2 *The* Haar measure *of a Sturm–Liouville hypergroup* $(\mathbb{R}_+, *)$ *with admissible function* A *is of the form*

$$\omega := A \cdot \lambda_{\mathbb{R}_+},$$

where $\lambda_{\mathbb{R}_+}$ *denotes the Lebesgue measure restricted to* $\mathbb{R}_+$.

In fact, the admissible function A is unique.

7.2.9.3 *Let* A *be an admissible function such that there exists a function* $\beta \in C^1(\mathbb{R}_+)$ *with the properties that* $\beta(0) \geqslant 0$ *and that the functions*

$$\frac{A'}{A} - \beta$$

and

$$\beta' - \frac{\beta^2}{2} + \beta \frac{A'}{A}$$

are decreasing. Then there exists a Sturm–Liouville hypergroup $(\mathbb{R}_+, *_A)$ *whose convolution* $*$ *depending on* A *and* β *satisfies*

$$\operatorname{supp}(\varepsilon_x * \varepsilon_y) \subset \big[|x - y|, x + y\big]$$

for all $x, y \in \mathbb{R}_+$.

7.2.9.4 *The* generalized translation operator *T^x associated with a Sturm–Liouville hypergroup* $(\mathbb{R}_+, *_A)$ *commutes with the (Sturm–Liouville) operator* L_A *in the sense that*

$$T^x L_A f = L_A T^x f$$

for all $f \in C^2(\mathbb{R}_+)$ *with* $f'(0) = 0$ *and* $x \in \mathbb{R}_+$.

7.2.9.5 *Let* $(\mathbb{R}_+, *_A)$ *be a Sturm–Liouville hypergroup satisfying the assumptions of Property 7.2.9.3.*

By

$$\varrho := \frac{1}{2} \lim_{x \to \infty} \frac{A'}{A}(x) \geqslant 0$$

we denote the **index** *of the hypergroup* $(\mathbb{R}_+, *_A)$.

The multiplicative functions χ on $(\mathbb{R}_+, *_A)$ are exactly the solutions of the eigenvalue problem

$$L_A \chi = s_\chi \chi$$

with $\chi(0) = 1$ *and* $\chi'(0) = 0$, *for some* $s_\chi \in \mathbb{C}$.

But then the dual $\mathbb{R}_+^\wedge$ *coincides with the set*

$$\{\chi \in \mathbb{R}_+^* \cap C^2(\mathbb{R}_+) : s_\chi \in \mathbb{R}_+\}$$

or after reparametrization via

$$\lambda_\chi^2 + \varrho^2 = s_\chi\,,$$

where $\lambda_\chi \in \mathbb{C}$ *with* $\operatorname{Im}\lambda_\chi \geqslant 0$, *with the set*

$$\{\chi \in \mathbb{R}_+^* \cap C^2(\mathbb{R}_+) : \lambda_\chi \in \mathbb{R}_+ \cup \mathrm{i}\,]0, \varrho]\}\,.$$

Moreover,

$$\mathbb{R}_+^\sim = \{\chi \in \mathbb{R}_+^\wedge : \lambda_\chi \in \mathbb{R}_+\}\,.$$

We shall identify $\mathbb{R}_+^\wedge$ and $\mathbb{R}_+^\sim$ with $\mathbb{R}_+ \cup \mathrm{i}\,]0, \varrho]$ and $\mathbb{R}_+$ respectively.

It is to be observed that for general Sturm–Liouville hypergroups the Plancherel measure π associated with the Haar measure ω cannot easily be computed. For special cases explicit computations will be taken up later.

Examples 7.2.10 of Sturm–Liouville hypergroups.

7.2.10.1 For

$$A(x) := x^{2\alpha+1}$$

whenever $x \in \mathbb{R}_+$, with $\alpha > -\frac{1}{2}$ one obtains the **Bessel–Kingman hypergroup** $(\mathbb{R}_+, *_\alpha)$ whose convolution is given by

$$\varepsilon_x *_\alpha \varepsilon_y(\mathrm{d}z) = K_\alpha(x, y, z)\, x^{2\alpha+1}\, \lambda_{\left[|x-y|, x+y\right]}(\mathrm{d}z)\,,$$

where

$$K_\alpha(x, y, z) := \frac{\Gamma(\alpha+1)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha+\frac{1}{2}\right)2^{2\alpha-1}} \frac{\left[(z^2 - (x-y)^2)\left((x+y)^2 - z^2\right)\right]^{\alpha-\frac{1}{2}}}{(xyz)^{2\alpha}}$$

for all $x, y, z \in \mathbb{R}_+^\times$.

The characters of $(\mathbb{R}_+, *_\alpha)$ of the form φ_λ with $\lambda \in \mathbb{R}_+$ are given by

$$\varphi_\lambda(x) := J_\alpha(\lambda x)$$

for all $x \in \mathbb{R}_+$, where J_α denotes the (complex-valued) modified Bessel function of order α given by

$$J_\alpha(z) := \sum_{k \geqslant 0} \frac{(-1)^k\, \Gamma(\alpha+1)}{2^{2k}\, k!\, \Gamma(\alpha+k+1)} z^{2k}$$

for all $z \in \mathbb{C}$. It turns out that

$$\mathbb{R}_+^\wedge = \mathbb{R}_+$$

and since $\varrho = 0$,

$$\mathbb{R}_+^\wedge = \mathbb{R}_+^\sim .$$

In particular, the Bessel–Kingman hypergroup $(\mathbb{R}_+, *_\alpha)$ is selfdual, hence Pontryagin, and the Plancherel measure π is proportional to the Haar measure ω.

For $\alpha := \frac{d}{2} - 1$, $d \geqslant 2$,

$$\mathbb{R}_+ \cong \mathbb{M}(d)//\mathrm{SO}(d)$$

appears as the *motion group* quoted in 7.1.5.3 (2.1).

7.2.10.2 Let $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ with $\alpha \neq -\frac{1}{2}$. Then the function

$$x \mapsto A(x) := \sinh^{2\alpha+1} x \, \cosh^{2\beta+1} x$$

on $\mathbb{R}_+$ defines the **Jacobi hypergroup** (of noncompact type) $(\mathbb{R}_+, *_{\alpha,\beta})$. Its index is

$$\varrho := \alpha + \beta + 1 \,,$$

and the characters are the Jacobi functions which can be identified with the elements of the dual space

$$\mathbb{R}_+^\wedge = \mathbb{R}_+ \cup \mathrm{i}\,]0, \varrho]$$

of $(\mathbb{R}_+, *_{\alpha,\beta})$.

For Jacobi hypergroups the Plancherel measure (associated with the Haar measure ω) has been computed as

$$\pi(\mathrm{d}\lambda) = \frac{1}{|c(\lambda)|^2} \lambda_{\mathbb{R}_+}(\mathrm{d}x)$$

with

$$c(\lambda) = c_{\alpha,\beta}(\lambda) := \frac{\Gamma(\alpha+1)\,\Gamma\left(\frac{\mathrm{i}\lambda}{2}\right)\Gamma\left(\frac{1+\mathrm{i}\lambda}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{\alpha+\beta+1+\mathrm{i}\lambda}{2}\right)\Gamma\left(\frac{\alpha-\beta+1+\mathrm{i}\lambda}{2}\right)}$$

$$= \frac{\sqrt{2\pi}\,2^{-\mathrm{i}\lambda}\,\Gamma(\mathrm{i}\lambda)\,\Gamma(\alpha+1)}{\Gamma\left(\frac{\varrho+\mathrm{i}\lambda}{2}\right)\Gamma\left(\frac{\varrho+\mathrm{i}\lambda}{2} - \beta\right)}$$

for all $\lambda \in \mathbb{R}_+$.

For $\beta = -\frac{1}{2}$, hence $\varrho = \alpha + \frac{1}{2}$, we obtain **hyperbolic hypergroups**. Further specialization to $\alpha := \frac{1}{2}(d-2)$ for $d \geqslant 2$ yields **hyperbolic hypergroups of rank** 1 which are double coset hypergroups of the form

$$\mathrm{SO}_0(d,1)//\mathrm{SO}(d) \,.$$

(See 7.1.5.3 (2.2).)

In this context we quote an important structural result as

Theorem 7.2.11 *Let G be a locally compact group and H a compact subgroup of G such that the double coset hypergroup $G//H$ is homeomorphic to $\mathbb{R}_+$. Then $(G//H, *)$ is a Sturm–Liouville hypergroup $(\mathbb{R}_+, *_A)$ of noncompact type, and the corresponding 5 types of admissible functions A contain those of Examples 7.2.10.1 and 7.2.10.2.*

(See [10], 3.5.89.)

II Convolution semigroup of measures

We are now ready to study one-parameter families of measures on an arbitrary commutative hypergroup $(K, *)$ and to step herewith into probability theory on hypergroups.

Definition 7.2.12 A family $(\mu_t)_{t \geqslant 0}$ of measures in $M^1(K)$ is called a (continuous) *convolution semigroup* on K if

$$\mu_s * \mu_s = \mu_{s+t} \quad \text{for all } s, t \geqslant 0,$$
$$\mu_0 = \varepsilon_0$$

and if the mapping

$$t \mapsto \mu_t$$

from $\mathbb{R}_+$ into $M^1(K)$ is continuous, where $M^1(K)$ carries the weak topology τ_{w}.

In order to represent convolution semigroups on K in terms of (generating) functions on $K^\wedge$ we recall the notions of positive and negative definiteness for Abelian groups which carry over to commutative hypergroups. For the duals of commutative hypergroups we need stronger notions.

Let

$$M_{\mathrm{ac}}(K^\wedge) := \{\mu = c\varepsilon_\mathbb{1} + g\pi_K \in M^{\mathrm{b}}(K^\wedge) : c \in \mathbb{C}, \ g \in C^{\mathrm{c}}(K^\wedge)\}$$

Definition 7.2.13 Let $\psi \in C(K^\wedge)$.

(1) ψ is said to be **strongly positive definite** if

$$\int_{K^\wedge} \psi \, \mathrm{d}\mu \geqslant 0$$

holds for every $\mu \in M_{\mathrm{ac}}(K^\wedge)$ with $\check{\mu} \geqslant 0$.

(2) ψ is called **strongly negative definite** if $\psi(\mathbb{1}) \geqslant 0$,

$$\psi(\overline{\chi}) = \overline{\psi(\chi)}$$

for all $\chi \in K^{\wedge}$ and

$$\int_{K^{\wedge}} \psi \, \mathrm{d}\mu \leqslant 0$$

whenever $\mu \in M_{\mathrm{ac}}(K^{\wedge})$ with $\check{\mu} \geqslant 0$ and $\check{\mu}(e) = 0$.

The sets of strongly positive and strongly negative definite functions on $K^{\wedge}$ will be abbreviated by $\mathrm{SPD}(K^{\wedge})$ and $\mathrm{SND}(K^{\wedge})$ respectively.

Remark 7.2.14 The strong notions of definiteness are motivated by the fact that they do not depend on the behavior of the functions outside $K^{\sim} \setminus \{\mathbb{1}\}$, hence the Bochner and Schoenberg representations (to be quoted below) can only be expected to hold on the regular dual $K^{\sim}$.

Clearly, for strong hypergroups K the strong notions coincide with the usual ones. (See [10], 4.1.1 and 4.4.1.)

Theorem 7.2.15 (of Bochner for $K^{\wedge}$)
For every $\psi \in \mathrm{SPD}(K^{\wedge})$ there exists a unique measure $\mu \in M_{+}^{\mathrm{b}}(K)$ such that

$$\widehat{\mu} = \psi \quad \pi_K\text{-locally almost surely.}$$

If, in addition, $\psi \in C(K^{\wedge})$, then

$$\mathrm{Res}_{K^{\sim}} \widehat{\mu} = \mathrm{Res}_{K^{\sim}} \psi \, .$$

Conversely, given $\mu \in M_{+}^{\mathrm{b}}(K)$ the Fourier transform $\widehat{\mu}$ of μ always belongs to $\mathrm{SPD}(K^{\wedge})$.

Theorem 7.2.16 (Schoenberg correspondence)
For every convolution semigroup $(\mu_t)_{t \geqslant 0}$ on K there exists a unique $\psi \in \mathrm{SND}(K^{\wedge})$ with $\psi(\mathbb{1}) = 0$ which satisfies

$$\widehat{\mu_t} = \exp(-t\psi)$$

for all $t \geqslant 0$.

Conversely, given a $\psi \in \mathrm{SND}(K^{\wedge})$ with $\psi(\mathbb{1}) = 0$, there exists a unique convolution semigroup $(\mu_t)_{t \geqslant 0}$ on K such that

$$\mathrm{Res}_{K^{\sim}} \widehat{\mu} = \mathrm{Res}_{K^{\sim}} \exp(-t\psi)$$

for all $t \geqslant 0$.

Canonical decompositions of convolution semigroups on commutative hypergroups K can be given in terms of Poisson and Gaussian semigroups ([64]) or in terms of their strongly negative definite functions on $K^\wedge$ ([10], 4.5).

In the following we shall discuss the first mentioned type of canonical decompositions.

Definition 7.2.17 A **symmetric abstract Lévy measure** $\lambda \in M_+(K)$ is defined by the Fourier transform

$$\chi \mapsto \exp\left(\int (\operatorname{Re}\chi - 1)\,\mathrm{d}\lambda\right)$$

of a **generalized Poisson measure** $\mathrm{e}(\lambda)$ (related to λ) $\in M^1(K)$. In general the integral in the Fourier transform need not exist as a real number. If, however,

$$\int (\operatorname{Re}\chi - 1)\,\mathrm{d}\lambda < \infty$$

for all $\chi \in K^\wedge$, then the family $\big(\mathrm{e}(t\lambda)\big)_{t \geqslant 0}$ forms a **generalized Poisson convolution semigroup** on K.

On the other hand one introduces the exponential of a measure $\mu \in M_+^{\mathrm{b}}(K)$ by

$$\exp(\mu) := \sum_{k \geqslant 0} \frac{1}{k!}\mu^k$$

with $\mu^0 := \varepsilon_e$. If $\mu(\{e\}) = 0$, then μ is an **abstract Lévy measure** in the sense that $\mu + \mu^-$ is a symmetric abstract Lévy measure, and

$$\mathrm{e}(\mu) = \mathrm{e}^{-\|\mu\|} \exp(\mu).$$

With this definition of $\mathrm{e}(\mu)$ for arbitrary $\mu \in M_+^{\mathrm{b}}(K)$ the measure $\mathrm{e}(\mu)$ becomes the **Poisson measure** and $\big(\mathrm{e}(t\mu)\big)_{t \geqslant 0}$ the **Poisson semigroup** with exponent μ respectively.

Given an arbitrary convolution semigroup $(\mu_t)_{t \geqslant 0}$ on K, there exists a unique measure $\lambda \in M_+(K^\wedge)$ such that

$$\lambda = \tau_{\mathrm{v}}\text{--}\lim_{t \to 0} \frac{1}{t} \operatorname{Res}_{K^\times} \mu_t.$$

λ is called the **Lévy measure** of the convolution semigroup $(\mu_t)_{t \geqslant 0}$ provided $\lambda(\{e\}) = 0$ and

$$\tau_{\mathrm{v}}\text{--}\lim_{t \to 0} \frac{1}{t} \operatorname{Res}_{K^\times} \mu_t = \operatorname{Res}_{K^\times} \lambda.$$

Now, the definition of a **Gaussian convolution semigroup** $(\nu_t)_{t \geqslant 0}$ by the requirement

$$\lim_{t \to 0} \frac{1}{t} \nu_t(\complement U) = 0$$

for every open set U of K with $e \in K$ is well motivated.

Gaussian measures on K are introduced as measures $\nu \in M^1(K)$ with the property that there exists a Gaussian convolution semigroup $(\nu_t)_{t \geqslant 0}$ on K with $\nu_1 = \nu$.

It can be shown that Gaussian convolution semigroups are exactly the convolution semigroups on K with Lévy measure $\lambda = 0$.

The proof of this equivalence makes use of the notion of the **(infinitesimal) generator** L of a convolution semigroup $(\mu_t)_{t \geqslant 0}$ on K defined by

$$L\varphi := \tau_{\mathrm{co}} - \lim_{t \to 0} \frac{1}{t} (\mu_t * \varphi - \varphi)$$

for all φ in the domain

$$D(L) := \left\{ \varphi \in C(K) : \tau_{\mathrm{co}} - \lim_{t \to 0} \frac{1}{t} (\mu_t * \varphi - \varphi) \ \text{ exists} \right\}$$

of L.

It turns out that a convolution semigroup $(\mu_t)_{t \geqslant 0}$ with generator L is Gaussian if and only if L is **local** in the sense that

$$\operatorname{supp} L f \subset \operatorname{supp} f$$

for all $f \in D(L) \cap C^{\mathrm{c}}(K)$.

Theorem 7.2.18 (Canonical decomposition of convolution semigroups) *Let K be a second countable commutative hypergroup and let $(\mu_t)_{t \geqslant 0}$ be a convolution semigroup on K with a symmetric Lévy measure λ. Then*

(i) *λ is a symmetric abstract Lévy measure with*

$$\int (1 - \operatorname{Re} \chi) \, \mathrm{d}\lambda < \infty$$

for all $\chi \in K^{\wedge}$, and $\big(\mathrm{e}(t\lambda)\big)_{t \geqslant 0}$ is a generalized Poisson semigroup on K.

(ii) *There exists a Gaussian semigroup $(\nu_t)_{t \geqslant 0}$ on K such that*

$$\mu_t = \nu_t * \mathrm{e}(t\lambda)$$

for all $t \geqslant 0$.

Remark 7.2.19 If the Lévy measure λ of the convolution semigroup $(\mu_t)_{t\geqslant 0}$ in the theorem is not symmetric, but satisfies the condition

$$\int |1 - \chi|\,d\lambda < \infty$$

for all $\chi \in K^\wedge$, then the decomposition in (ii) of the theorem remains valid.

Application 7.2.20 *Concrete canonical decompositions of convolution semigroups on a commutative hypergroup are given in terms of their associated strongly negative definite functions or generators. Concerning the correspondence of the latter objects we note that for every $\chi \in K^\wedge$*

$$L\chi = -\psi(\chi)\chi$$

on K.

We close this section with Schoenberg representations in the sense of Theorem 7.2.16 of convolution semigroups on the Jacobi polynomial hypergroups, the dual Jacobi hypergroups (7.2.5) and the Bessel–Kingman hypergroups (7.2.10.1).

7.2.20.1 *On the Jacobi polynomial hypergroup $(K,*) = (\mathbb{Z}_+, *_{\alpha,\beta})$ with $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ or $\alpha + \beta \geqslant 0$ any convolution semigroup $(\mu_t)_{t\geqslant 0}$ has a Lévy measure $\lambda \in M_+(\mathbb{Z}_+)$ which is symmetric and bounded. $(\mu_t)_{t\geqslant 0}$ admits a Schoenberg representation of the form*

$$\widehat{\mu_t} = \exp(-t\psi) \qquad (t \geqslant 0)$$

on $(K^\wedge, \widehat{*}) = \big([-1,1], \widehat{*}_{\alpha,\beta}\big)$ with

$$\psi(x) := \sum_{n\geqslant 1}(1 - Q_n^{(\alpha,\beta)}(x))\,\lambda(\{n\})$$

whenever $x \in [-1, 1]$.

7.2.20.2 *Convolution semigroups $(\mu_t)_{t\geqslant 0}$ on a dual Jacobi polynomial hypergroup $(K^\wedge, \widehat{*}) = ([-1,1], *_{\alpha,\beta})$, with Lévy measure $\lambda \in M_+([-1,1])$ admit a Schoenberg representation of the form*

$$\widehat{\mu_t} = \exp(-t\psi) \qquad (t \geqslant 0)$$

on $(K,*) = (\mathbb{Z}_+, *_{\alpha,\beta})$, where

$$\psi(n) := q(n) + \int_{-1}^{1-0}(1 - Q_n^{(\alpha,\beta)}(x))\,\lambda(dx)$$

with

$$q(n) := a\frac{n(n + \alpha + \beta + 1)}{\alpha + \beta + 2} \qquad (a \geqslant 0)$$

for all $n \in \mathbb{Z}_+$.

7.2.20.3 *If* $(\mu_t)_{t \geqslant 0}$ *is a convolution semigroup on the Bessel–Kingman hypergroup* $(\mathbb{R}_+, *_\alpha)$ $(\alpha > -\frac{1}{2})$ *with Lévy measure* $\lambda \in M_+(\mathbb{R}_+^\times)$, *then its Schoenberg correspondence reads as*

$$\widehat{\mu_t} = \exp(-t\psi) \qquad (t \geqslant 0)$$

with

$$\psi(\xi) := c\xi^2 + \int_{\mathbb{R}_+^\times} (1 - \varphi_\xi(x)) \, \lambda(\mathrm{d}x) \qquad (c \geqslant 0)$$

for all $\xi \in \mathbb{R}_+$.

There are similar representations of convolution semigroups on higher-dimensional hypergroups such as polynomial hypergroups in several variables ([100], [11]), local Sturm–Liouville hypergroups, disc hypergroups, two-dimensional Jacobi hypergroups, Laguerre hypergroups ([65]) and expected desirable extensions to matrix–cone hypergroups ([73], [92]).

Discussion 7.2.21 More on Gaussian convolution semigroups on a Sturm–Liouville hypergroup $(\mathbb{R}_+, *_A)$.

At first we note that the Sturm–Liouville operator $-L_A$ is the generator of a Gaussian convolution semigroup $(\nu_t)_{t \geqslant 0}$ on $\mathbb{R}_+$, and that each Gaussian convolution semigroup $(\nu_t)_{t \geqslant 0}$ on $\mathbb{R}_+$ is of the form

$$\mu_t = \nu_{ct} \qquad (t \geqslant 0)$$

for some $c \geqslant 0$, with a Schoenberg representation of the form

$$\widehat{\mu_t}(\xi) = \exp(-ct(\xi^2 + \varrho^2)) \qquad (t \geqslant 0)$$

whenever $\xi \in \mathbb{R}_+ \cup \mathrm{i}]0, \varrho]$. If $c = 1$, then the Gaussian convolution semigroup $(\nu_t)_{t \geqslant 0}$ on $\mathbb{R}_+$ is called **normalized**.

For each $t > 0$ the measure ν_t appears as the **normalized Gaussian measure** with parameter t.

In case of a Bessel–Kingman hypergroup $(\mathbb{R}_+, *_\alpha)$ with $\alpha \geqslant -\frac{1}{2}$ the normalized convolution semigroup $(\gamma_t^{(\alpha)})_{t \geqslant 0}$ can be given explicitly by

$$\gamma_t^{(\alpha)}(\mathrm{d}x) = \frac{1}{2^{2\alpha+1}\,\Gamma(\alpha+1)} \frac{1}{t^{\alpha+1}} x^{2\alpha+1} \mathrm{e}^{-\frac{x^2}{4t}} \, \mathrm{d}x \qquad (t > 0).$$

For $t = \frac{1}{2}$, $\gamma_t^{(\alpha)}$ is called the **Rayleigh distribution** on $\mathbb{R}_+$ which in the special case $\alpha := \frac{d}{2} - 1$ can be interpreted as the radial part of the normal distribution on $\mathbb{R}^d$.

Any Gaussian process associated with the **Bessel semigroup** $(\beta_t^{(\alpha)})_{t \geqslant 0}$ with

$$\beta_t^{(\alpha)} := \gamma_{\frac{t}{2}}^{(\alpha)}$$

for all $t > 0$ is called a **Bessel process** of order $\alpha \geqslant -\frac{1}{2}$. Note that the radial parts of d-dimensional Brownian motions are Bessel processes of order $\alpha = \frac{d}{2} - 1$ in the traditional sense.

Remark 7.2.22 Convolution semigroups on an arbitrary commutative hypergroup enjoy a wide range of applications which can hardly be indicated in the present exposé. We refer the reader to the problem of embedding infinitely divisible probability measures into convolution semigroups and to the rôle that convolution semigroups play in the theory of potentials and Dirichlet forms ([10] 5.3, 6.4 and 6.5 respectively).

7.3   Random walks in hypergroups

This section is designated to discussing random walks in a commutative hypergroup. There are at least two different ways to introduce discrete time stochastic processes: the purely measure-theoretic one in terms of Markov kernels and the probabilistic one which makes use of a randomized summation of hypergroup-valued random variables. The construction of randomized sums will depend on a concretization of the underlying hypergroup.

I   Transient random walks

Let K be an arbitrary hypergroup.

Definition 7.3.1 A triplet (Ξ, ν, Φ) consisting of a compact space Ξ, a measure $\nu \in M^1(\Xi)$, and a Borel-measurable mapping

$$\Phi : K \times K \times \Xi \to K$$

is called a **concretization** of the hypergroup K if

$$\nu\big(\{z \in \Xi : \Phi(x, y, z) \in A\}\big) = (\varepsilon_x * \varepsilon_y)(A)$$

for all $x, y \in K$ and $A \in \mathfrak{B}(K)$.

It can be shown that for second countable hypergroups K there always exists a mapping

$$\Phi : K \times K \times [0, 1] \to K$$

such that $([0, 1], \lambda_{[0,1]}, \Phi)$ is a concretization of K.

Examples 7.3.2 of concretizations.

7.3.2.1 Let G be a locally compact group with (deterministic) multiplication and convolution $*$. Then the triplet (Ξ, ν, Φ) with

$$\Xi := \{e\},$$
$$\nu := \varepsilon_e$$

and

$$\Phi(x, y, e) := x \cdot y$$

for all $x, y \in G$ is a concretization of G.

7.3.2.2 Let G be a locally compact group and H a compact subgroup of G. We consider the *double coset hypergroup* $K := G//H$. The triplet (Ξ, ν, Φ) with

$$\Xi := H\,,$$
$$\nu := \omega_H$$

and

$$\Phi(x, y, h) := H\,\phi(x)\,h\,\phi(y)\,H$$

for all $x, y \in K$, $h \in H$ and a measurable mapping $\phi : K \to G$ satisfying

$$H\,\phi(x)\,H = x$$

for all $x \in K$ is a concretization of k.

7.3.2.3 A concretization of the *Bessel–Kingman hypergroup* $(\mathbb{R}_+, *_\alpha)$ (7.2.10.1) is given by the triplet (Ξ, ν, Φ), where

$$\Xi := [-1, 1]\,,$$
$$\nu := g \cdot \lambda_\Xi$$

with

$$g(\xi) := c_\alpha (1 - \xi^2)^{\alpha - \frac{1}{2}}$$

for all $\xi \in \Xi$, and

$$\Phi(x, y, \xi) := \sqrt{x^2 + y^2 - 2\xi x y}$$

whenever $x, y \in \mathbb{R}_+$, $\xi \in \Xi$.

7.3.2.4 For the *Jacobi hypergroup* $(\mathbb{R}_+, *_{\alpha,\beta})$ with $\alpha > \beta > -\frac{1}{2}$ we have a concretization (Ξ, ν, Φ), where

$$\Xi := [0, 1] \times [0, \pi]\,,$$
$$\nu := g \cdot \lambda_\Xi^2$$

with

$$g(r, \vartheta) := c_{\alpha,\beta}\,(1 - r^2)^{\alpha - \beta - 1}\,r^{2\beta + 1}\,\sin^{2\beta}\vartheta$$

for all $r \in [0, 1]$, $\vartheta \in [0, \pi]$ and

$$\Phi(x, y, (r, \vartheta)) := \frac{1}{2}\operatorname{arcosh}\left(2\cosh^2 x\,\cosh^2 y + 2r^2 \sinh^2 x\,\sinh^2 y\right.$$

$$\left. + r\cos\vartheta\,\sinh(2x)\,\sinh(2y) - 1\right)$$

whenever $x, y \in \mathbb{R}_+$, $(r, \vartheta) \in \Xi$.

Employing the concretization of the hypergroup K we introduce the randomized sum of K-valued random variables.

Let (Ξ, ν, Φ) denote a fixed concretization of K.

Definition 7.3.3 For any two K-valued random variables X and Y on a probability space $(\Omega, \mathfrak{A}, \mathsf{P})$ and an auxiliary Ξ-valued random variable Λ on $(\Omega, \mathfrak{A}, \mathsf{P})$ such that Λ is (stochstically) independent of the $(K \times K)$-valued random variable $X \otimes Y$ and distributed with law ν we define the **randomized sum** of X and Y by

$$X \stackrel{\Lambda}{+} Y := \Phi(X, Y, \Lambda)\,.$$

Clearly, $X \stackrel{\Lambda}{+} Y$ is a K-valued random variable on $(\Omega, \mathfrak{A}, \mathsf{P})$.

The desired consistency of the randomized sum of K-valued random variables with the convolution of the hypergroup K is justified by the following important

Fact 7.3.4 *Given K-valued random variables X and Y and a Ξ-valued random variable Λ on $(\Omega, \mathfrak{A}, \mathsf{P})$ with law ν such that X, Y and Λ are independent we have*

$$\mathsf{P}_{X \stackrel{\Lambda}{+} Y} = \mathsf{P}_X * \mathsf{P}_Y\,.$$

We note that forming randomized sums is generally not an associative operation although convolution of distributions is. While the randomized sum $X \stackrel{\Lambda}{+} Y$ depends on the choice of the applied concretization of K the joint distribution of the random variables X, Y and $X \stackrel{\Lambda}{+} Y$ does not.

Now we are ready to introduce random walks in K and to describe them by means of randomized sums of K-valued random variables.

We return to the assumption that K is a second countable *commutative* hypergroup.

Definition 7.3.5 A sequence $(S_n)_{n \geqslant 0}$ of K-valued random variables on a probability space $(\Omega, \mathfrak{A}, \mathsf{P})$ is said to be a **nonhomogeneous random walk** in K with law sequence $(\mu_n)_{n \geqslant 0}$ in $M^1(K)$ if it forms a nonhomogeneous Markov chain with starting measure μ_0 and transition probabilities given by

$$\mathsf{P}(S_n \in A \mid S_{n-1} = x) = (\mu_n * \varepsilon_x)(A) \qquad (n \geqslant 1)$$

whenever $A \in \mathfrak{B}(K)$ and $x \in K$.

For a law sequence $(\mu_n)_{n \geqslant 0}$ with $\mu_n := \mu \in M^1(K)$ for all $n \geqslant 1$ the sequence $(S_n)_{n \geqslant 0}$ is called a (homogeneous) **random walk** in K with law μ.

For the two types of random walks in K we apply the notation $X((\mu_n))$ and $X(\mu)$ respectively.

Fact 7.3.6 *Evidently, for any random walk $X((\mu_n))$ in K*

$$\mathsf{P}_{S_n} = \underset{j=0}{\overset{n}{\ast}} \mu_j$$

for all $n \geqslant 1$.

It should be noted that given a sequence $(\mu_n)_{n \geqslant 0}$ of measures in $M^1(K)$ the associated sequence $(P_n)_{n \geqslant 0}$ of Markov kernels defined by

$$P_n(x, A) := \mu_n \ast \varepsilon_x(A) \qquad (n \geqslant 0)$$

for all $x \in K$, $A \in \mathfrak{B}(K)$ generates a canonical Markov chain $\big(\Omega, \mathfrak{A}, \mathsf{P}^\nu, (S_n)_{n \geqslant 0}\big)$, where

$$\mathfrak{A} := \sigma \left(\bigcup_{n \geqslant 0} \sigma(S_0, S_1, \ldots, S_n) \right)$$

and P^ν is a probability measure on $(\Omega, \mathfrak{A})$ for every starting measure $\nu \in M^1(K)$. In fact,

$$\mathsf{P}^\nu(S_0 \in A) = \nu(A)$$

for all $A \in \mathfrak{A}$.

We adopt the convention P^x for $\mathsf{P}^{\varepsilon_x}$ $(x \in K)$.

Construction 7.3.7 of a random walk $X((\mu_n))$ in K in terms of randomized sums.

Let (Ξ, ν, Φ) be a concretization of K. Let $(X_n)_{n \geqslant 0}$ be a sequence of independent K-valued random variables on a probability space $(\Omega, \mathfrak{A}, \mathsf{P})$ with $\mathsf{P}_{X_n} = \mu_n$ for all $n \geqslant 0$, and let $(\Lambda_n)_{n \geqslant 1}$ be a sequence of independent Ξ-valued random variables on a probability space $(\Omega', \mathfrak{A}', \mathsf{P}')$ with $\mathsf{P}'_{\Lambda_n} = \nu$ for all $n \geqslant 1$. Suppose that the random variables constituting the sequences $(X_n)_{n \geqslant 0}$ and $(\Lambda_n)_{n \geqslant 1}$ are all independent. Then the sequence $(S_n)_{n \geqslant 0}$ defined by

$$S_0 := X_0 \,,$$
$$S_1 := X_0 \overset{\Lambda_1}{+} X_1 \,,$$

and

$$S_n := S_{n-1} \overset{\Lambda_n}{+} X_n$$

is a random walk $X((\mu_n))$ in K. In the applications we shall employ the notation

$$S_n =: \sum_{j=0}^{n}{}^{\Lambda} X_j \qquad (n \geqslant 0).$$

From the canonical construction of Markov chains we know that one can consider the random variables X_n as the coordinate maps on the probability space $(\Omega, \mathfrak{A}, \mathsf{P})$ with $\Omega := K^{\mathbb{Z}_+}$, $\mathfrak{A} := \bigotimes_{j \geqslant 0} \mathfrak{B}(K)$ and $\mathsf{P} := \bigotimes_{j \geqslant 0} \mu_j$, and analogously we can look at the random variables Λ_n as the coordinate maps on $(\Omega', \mathfrak{A}', \mathsf{P}')$ with $\Omega' := \Xi^{\mathbb{N}}$, $\mathfrak{A}' := \bigotimes_{j \geqslant 1} \mathfrak{B}(\Xi)$ and $\mathsf{P}' := \bigotimes_{j \geqslant 1} \nu$. Hence for each $n \geqslant 0$ the K-valued random variables S_n are defined on the probability space $(\Omega \times \Omega', \mathfrak{A} \otimes \mathfrak{A}', \mathsf{P} \otimes \mathsf{P}')$, and for $\boldsymbol{\omega} = (\omega, \omega') \in \Omega \times \Omega'$ we have

$$S_n(\boldsymbol{\omega}) = \left(X_0(\boldsymbol{\omega}) \overset{\Lambda_1(\omega')}{+} X_1(\boldsymbol{\omega}) \overset{\Lambda_2(\omega')}{+} \cdots \overset{\Lambda_{n-1}(\omega')}{+} X_{n-1}(\boldsymbol{\omega}) \right) \overset{\Lambda_n(\omega')}{+} X_n(\boldsymbol{\omega}).$$

The subsequent discussion will be devoted to the asymptotic behavior of random walks in K. We start with the properties of recurrence and transience and proceed to establishing two pearls of probability theory within the framework of hypergroups: strong law of large numbers and central limit theorems.

Given a random walk $X(\mu)$ in K and $A \in \mathfrak{B}(K)$ we introduce the function h_A on K defined by

$$h_A(x) := \mathsf{P}^x(S_n \in A \text{ infinitely often})$$

$$= \mathsf{P}^x \left(\sum_{n \geqslant 0} \mathbb{1}_A \circ S_n = \infty \right)$$

for all $x \in K$.

Definition 7.3.8 An element $x \in K$ is said to be **recurrent** if for every neighbourhood $V \in \mathfrak{V}(x)$

$$h_V(x) = 1,$$

and it is called **transient** if there exists a neighbourhood $W \in \mathfrak{V}(x)$ such that

$$h_W(x) = 0.$$

We note that a random walk $X(\mu)$ in K is called **adapted** provided

$$[\operatorname{supp}\mu]^- = K\,,$$

and **spread-out** if for some $p \in \mathbb{N}$, some constant $c > 0$ and some open subset $V \neq \emptyset$ of K we have

$$\mu^p \geqslant c\,\mathbb{1}_V\,\omega_K\,.$$

Theorem 7.3.9 of dichotomy.
Let K be a second countable commutative hypergroup. For a random walk $X(\mu)$ in K with an adapted and spread-out law $\mu \in M^1(K)$ there are only two possibilities:

(1) *Every $x \in K$ is recurrent*

or

(2) *every $x \in K$ is transient.*

Moreover,

(1′) *Case (1) occurs if and only if $X(\mu)$ is **Harris recurrent** in the sense that for every $A \in \mathfrak{B}(K)$*

$$\omega_K(A) > 0 \quad implies \quad h_A \equiv 1$$

whereas

$$\omega_K(A) = 0 \quad implies \quad h_A \equiv 0\,.$$

(2′) *Case (2) occurs if and only if for every compact $C \in \mathfrak{B}(K)$ the function*

$$x \mapsto U(x,C) := \sum_{n\geqslant 0}(\mu^n * \varepsilon_x)(A)$$

on K is bounded.

Definition 7.3.10 A random walk $X(\mu)$ in a second countable commutative hypergroup K with law $\mu \in M^1(K)$ is said to be **transient** if every point $x \in K$ is transient in the sense of Definition 7.3.8.

A not necessarily spread-out random walk $X(\mu)$ is called **weakly transient** provided

$$U(e,C) < \infty$$

for every compact $C \in \mathfrak{B}(K)$.

Finally we call a hypergroup K **transient** if every random walk $X(\mu)$ in K with adapted and spread-out law $\mu \in M^1(K)$ is transient.

One observes that transience of random walks in K implies weak transience, and that the reverse implication holds if $X(\mu)$ is adapted and spread-out.

In order to test the transience of a hypergroup we need an analogue of the well-known Chung–Fuchs criterion for locally compact Abelian groups.

Theorem 7.3.11 *Let K be a second countable commutative hypergroup which is either discrete or strong, and let $X(\mu)$ be a random walk in K with law $\mu \in M^1(K)$. Then $X(\mu)$ is weakly transient if and only if there exists a neighbourhood $N \in \mathfrak{V}_{K^\wedge}(\mathbb{1})$ such that*

$$\limsup_{t\uparrow 1} \int_N \mathrm{Re}\,\frac{1}{1 - t\widehat{\mu}(\chi)}\, \pi_K(\mathrm{d}\chi) < \infty\,.$$

Corollary 7.3.12 *Under the assumptions on K as in the theorem a random walk $X(\mu)$ with symmetric law $\mu \in M^1(K)$ is weakly transient if and only if*

$$\int_N \frac{1}{1 - \widehat{\mu}(\chi)}\, \pi_K(\mathrm{d}\chi) < \infty$$

for some neighbourhood $N \in \mathfrak{V}_{K^\wedge}(\mathbb{1})$.

A sufficient condition for a hypergroup to be transient is provided by the following

Theorem 7.3.13 *Let K be a second countable commutative hypergroup such that $\mathbb{1} \notin K^\sim$. Then every random walk $X(\mu)$ with adapted law $\mu \in M^1(K)$ is weakly transient, hence transient.*

Examples 7.3.14 of transient hypergroups.

7.3.14.1 Let $(\mathbb{Z}_+, *_{(Q_n)})$ be a polynomial hypergroup, where the defining polynomial sequence $(Q_n)_{n\geqslant 0}$ satisfies the recurrence relations (1) in 7.2.1 (involving sequences $(a_n)_{n\in\mathbb{Z}_+}$, $(b_n)_{n\in\mathbb{Z}_+}$ and $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+^\times$, $\mathbb{R}_+$ and $\mathbb{R}_+^\times$ respectively).

The following statements are equivalent:

(i) $(\mathbb{Z}_+, *_{(Q_n)})$ is transient.

(ii) The simple random walk $X(\varepsilon_1)$ in $\mathbb{Z}_+$ is transient.

(iii) $\int_{-1}^1 \frac{1}{1-x}\, \pi_{\mathbb{Z}_+}(\mathrm{d}x) < \infty\,.$

(iv) $\sum_{n\geqslant 1} \frac{1}{a_n \omega_{\mathbb{Z}_+}(\{n\})} < \infty\,.$

7.3.14.2 Jacobi polynomial hypergroups $(\mathbb{Z}_+, *_{\alpha,\beta})$ (see 7.2.5) are transient if and only if $\alpha > 0$.

7.3.14.3 Let $(\mathbb{R}_+, *_A)$ be a Sturm–Liouville hypergroup with index ϱ. The following statements are equivalent:

(i) $(\mathbb{R}_+, *_A)$ is transient
(ii) $\int_0^1 \frac{1}{x^2} \pi_{\mathbb{R}_+}(\mathrm{d}x) < \infty$.

Moreover, Theorem 7.3.13 implies that $(\mathbb{R}_+, *_A)$ is transient provided $\varrho > 0$.

The notion of transience can be extended to convolution semigroups on a commutative hypergroup K. It gives rise to an extended potential theory.

Definition 7.3.15 A convolution semigroup $(\mu_t)_{t\geqslant 0}$ in $M^1(K)$ is said to be **transient** if

$$\int_0^\infty \int_K f \, \mathrm{d}\mu_t \, \mathrm{d}t < \infty$$

for any $f \in C_+^c(K)$. In this case the functional

$$f \mapsto \int_0^\infty \int_K f \, \mathrm{d}\mu_t \, \mathrm{d}t$$

on $C_+^c(K)$ defines the **potential measure** $\kappa \in M_+(K)$ of $(\mu_t)_{t\geqslant 0}$.

It is easy to see that for any $\mu \in M_+^b(K)$ the *Poisson semigroup* $(\mathrm{e}(t\mu))_{t\geqslant 0}$ with exponent μ is transient if and only if

$$\sum_{n\geqslant 0} \int_K f \, \mathrm{d}\mu^n < \infty$$

for all $f \in C_+^c(K)$. If $(\mu_t)_{t\geqslant 0}$ is transient, then the potential measure of $(\mathrm{e}(t\mu))_{t\geqslant 0}$ has the form

$$\kappa = \sum_{n\geqslant 0} \mu^n.$$

The connection between transience of convolution semigroups on and random walks in K becomes evident by the fact that an adapted and spread-out random walk $X(\mu)$ with law $\mu \in M^1(K)$ is transient if and only if the Poisson semigroup $(\mathrm{e}(t\mu))_{t\geqslant 0}$ with exponent μ is transient.

In order to present a hypergroup analogue of the famous classical Stone–Port criterion for transience of convolution semigroups on a commutative hypergroup K we rely on a **majorization property** (MP) which reads as follows:

(MP) For every $f \in C^{\mathrm{c}}_+(K)$ there exists a $g \in C^{\mathrm{c}}(K^{\wedge})$ such that $f \leqslant \check{g}$.

Clearly, the classes of discrete, compact, strong and dual hypergroups enjoy (MP).

Theorem 7.3.16 *Let K be a commutative hypergroup, and let $(\mu_t)_{t \geqslant 0}$ be a convolution semigroup on K admitting a Schoenberg representation in terms of a function $\psi \in \mathrm{SND}(K^{\wedge})$.*

(i) *If $(\mu_t)_{t \geqslant 0}$ is transient, then*

$$\mathrm{Re}\, \frac{1}{\psi} \in L^1_{\mathrm{loc}}(K^{\wedge}, \pi_K)\,.$$

(ii) *If K satisfies* (MP) *and*

$$\frac{1}{\psi} \in L^1_{\mathrm{loc}}(K^{\wedge}, \pi_K)\,,$$

then $(\mu_t)_{t \geqslant 0}$ is transient.

II Limit theorems for random walks

Stating strong laws of large numbers and central limit theorems for random walks in a second countable commutative hypergroup K requires the use of moments of K-valued random variables.

We assume to be given measurable locally bounded real-valued functions m_1 and m_2 on K satisfying the equalities

$$m_1(x * y) = m_1(x) + m_1(y)$$

and

$$m_2(x * y) = m_2(x) + m_2(y) + 2\, m_1(x)\, m_1(y)$$

valid for all $x, y \in K$; they are called **moment functions** for K of order one and two respectively.

For the discussion in this section we shall assume that

$$m_1^2 \leqslant m_2\,.$$

Definition 7.3.17 A measure $\mu \in M^1(K)$ is said to admit a **moment** of order $j = 1, 2$ if $m_j \in L^1(K, \mu)$. In this case we write

$$\mathsf{E}_*(\mu) := \int_K m_1 \, \mathrm{d}\mu$$

and

$$\mathsf{V}_*(\mu) := \int_K m_2 \, \mathrm{d}\mu - \mathsf{E}_*(\mu)^2 \geqslant 0$$

for the **modified expectation** and **variance** of μ respectively.

Similarly moments of order one and two are introduced for K-valued random variables X on a probability space $(\Omega, \mathfrak{A}, \mathsf{P})$ by

$$\mathsf{E}_*(X) := \mathsf{E}(m_1 \circ X)$$

and

$$\mathsf{V}_*(X) := \mathsf{E}(m_2 \circ X) - \mathsf{E}_*(X)^2 \geqslant 0$$

respectively.

Properties 7.3.18 of moments.

7.3.18.1 *Given a concretization* (Ξ, ν, Φ) *of* K, K-valued random variables X, Y *and a* Ξ-valued random variable Λ *such that* X, Y *and* Λ *are independent. Then*

(i) $\mathsf{E}_*(X \overset{\Lambda}{+} Y) = \mathsf{E}_*(X) + \mathsf{E}_*(Y)$ *provided* $\mathsf{E}_*(X)$ *and* $\mathsf{E}_*(Y)$ *exist.*

(ii) $\mathsf{V}_*(X \overset{\Lambda}{+} Y) = \mathsf{V}_*(X) + \mathsf{V}_*(Y)$.

7.3.18.2 *Let* $(S_n)_{n \geqslant 0}$ *be a random walk* $X((\mu_n))$ *in* K.

(i) *If* $\mathsf{E}_*(X_n)$ *exists for all* $n \geqslant 1$, *then*

$$\mathsf{E}(m_1 \circ S_n \mid S_{n-1}) = m_1 \circ S_{n-1} + \mathsf{E}_*(\mu_n) \qquad [\mathsf{P}]\,.$$

(ii) *If* $\mathsf{V}_*(X_n)$ *exists for all* $n \geqslant 1$, *then*

$$\mathsf{E}(v(S_n, \mathsf{E}_*(S_n)) \mid S_{n-1}) = v(S_{n-1}, \mathsf{E}_*(S_{n-1})) + \mathsf{V}_*(\mu_n) \qquad [\mathsf{P}]\,,$$

where

$$v(x, y) := m_2(x) - 2\,y\,m_1(x) + y^2$$

for all $x, y \in K$.

7.3.18.3 *Let* $(S_n)_{n \geqslant 0}$ *be a random walk* $X((\mu_n))$ *in* K.

(i) *If* $\mathsf{E}_*(\mu_n)$ *exists for all* $n \geqslant 0$, *then*

$$\bigl(m_1 \circ S_n - \mathsf{E}_*(S_n)\bigr)_{n \geqslant 0}$$

is a martingale, and

$$\bigl(m_1 \circ S_n\bigr)_{n \geqslant 0}$$

is a submartingale provided $\mathsf{E}_*(\mu_n) \geqslant 0$ *for all* $n \geqslant 0$.

(ii) *If* $V_*(\mu_n)$ *exists for all* $n \geqslant 0$, *then*

$$\left(v(S_n, E_*(S_n)) - V_*(S_n)\right)_{n \geqslant 0}$$

is a martingale, and

$$\left(v(S_n, E_*(S_n))\right)_{n \geqslant 0}$$

is a submartingale.

Property 7.3.18.3 implies immediately the following

Theorem 7.3.19 *Let* $(S_n)_{n \geqslant 0}$ *be a random walk* $X((\mu_n))$ *in* K *with law sequence* $(\mu_n)_{n \geqslant 0}$ *in* $M^1(K)$, *and let* $(r_n)_{n \geqslant 1}$ *be a sequence in* $\mathbb{R}_+^{\times}$ *with*

$$\lim_{n \to \infty} r_n = \infty.$$

Suppose that

$$\sum_{n \geqslant 1} \frac{V_*(\mu_n)}{r_n^2} < \infty.$$

Then

$$\frac{1}{r_n^2} v(S_n, E_*(S_n)) \to 0 \qquad [P].$$

If, in particular, $(S_n)_{n \geqslant 0}$ *is a random walk* $X(\mu)$ *with law* $\mu \in M^1(K)$ *such that* $V_*(\mu) < \infty$, *then*

$$\frac{1}{n} m_1 \circ S_n \to E_*(\mu) \qquad [P].$$

Examples 7.3.20 Choices of moment functions.

7.3.20.1 Let $(\mathbb{Z}_+, *_{(Q_n)})$ be a polynomial hypergroup, where the defining polynomial sequence $(Q_n)_{n \geqslant 0}$ satisfies the recurrence relation (1) in 7.2.1 (involving sequences $(a_n)_{n \in \mathbb{Z}_+}$, $(b_n)_{n \in \mathbb{Z}_+}$ and $(c_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_+^{\times}$, $\mathbb{R}_+$ and $\mathbb{R}_+^{\times}$ respectively). We assume that the sequence $(Q_n)_{n \geqslant 0}$ is normalized in the sense that $Q_1(x) = x$ for all $x \in [-1, 1]$, that the limits

$$\alpha := \lim_{n \to \infty} a_n \in]0, 1[,$$

$$\beta := \lim_{n \to \infty} b_n \in \mathbb{R},$$

$$\gamma := \lim_{n \to \infty} c_n \in]0, 1[$$

exist, and that the inequality $\alpha \geqslant \gamma$ holds.

A change of variables yields a new polynomial sequence $(R_n)_{n \geqslant 0}$ defined by

$$R_n(x) := Q_n(2\sqrt{\alpha\gamma}\, x + \beta)$$

for all $x \in \mathbb{R}$ with

$$R_n(x_0) = 1$$

for

$$x_0 := \frac{1 - \beta}{2\sqrt{\alpha\gamma}}\,.$$

Choosing the unique $\vartheta_0 \in i\mathbb{R}$ with $\cos\vartheta_0 = x_0$ the mappings

$$k \mapsto m_j(k) := i^j \left(\frac{d}{d\vartheta}\right)^j R_k(\cos\vartheta)\big|_{\vartheta=\vartheta_0}$$

from $\mathbb{Z}_+$ into $\mathbb{R}$ $(j = 1, 2)$ are moment functions for $(\mathbb{Z}_+, *_{(Q_n)})$.

For Jacobi hypergroups $(\mathbb{Z}_+, *_{\alpha,\beta})$ one calculates

$$m_2(k) = \frac{k(k + 1 + \alpha + \beta)}{2 + \alpha + \beta}$$

whenever $k \in \mathbb{Z}_+$.

7.3.20.2 Let $(\mathbb{R}_+, *_A)$ be a *Sturm–Liouville hypergroup* with defining Sturm–Liouville operator L_A and index $\varrho \geqslant 0$.

We know from 7.2.9.5 that for every $\lambda \in \mathbb{C}$ the solution φ_λ of the eigenvalue problem

$$L_A\varphi_\lambda = (\lambda^2 + \varrho^2)\varphi_\lambda$$

with $\varphi_\lambda(0) = 1$, $\varphi_\lambda'(0) = 0$ is a multiplicative function on $\mathbb{R}_+$ and that

$$\{\varphi_\lambda : \lambda \in \mathbb{R}_+ \cup i\,]0, \varrho]\} \cong \mathbb{R}_+ \cup i\,]0, \varrho]$$

forms the dual $\mathbb{R}_+^{\wedge}$ of $\mathbb{R}_+$, the unit character $\varphi_{i\varrho}$ being identified with $i\varrho$.

It can be shown that the mappings

$$x \mapsto m_j(x) := \left(\frac{d}{dt}\right)^j \varphi_{i(\varrho+t)}(x)\big|_{t=0}$$

from $\mathbb{R}_+$ into $\mathbb{R}$ $(j = 1, 2)$ are moment functions for $(\mathbb{R}_+, *_A)$. Moreover, $m_2 \geqslant 0$, $m_1 \equiv 0$ if $\varrho = 0$, and $m_1(x) \sim x$, $m_2(x) \sim x^2$ (as $x \to \infty$) if $\varrho > 0$.

m_1 and m_2 can be represented in closed form as

$$m_1(x) = 2\varrho \int_0^x \frac{1}{A(y)} \int_0^y A(z)\,dz\,dy$$

and

$$m_2(x) = 8\varrho^2 \int_0^x \int_0^y \int_0^z \int_0^u \frac{A(z)\,A(v)}{A(y)\,A(u)} \, dy\,dz\,du\,dv$$
$$+ 2 \int_0^x \int_0^y \frac{A(z)}{A(y)} \, dy\,dz$$

respectively, whenever $x \in \mathbb{R}_+$.

The connection between m_1 and m_2 becomes apparent from the formula

$$m_2(x) = \int_0^x \left(\int_u^x \frac{dz}{A(z)} \right) (2 + 4\,\varrho\, m_1(u)) A(u)\,du$$

valid for all $x \in \mathbb{R}_+$.

In the case of a *Bessel–Kingman hypergroup* $(\mathbb{R}_+, *_\alpha)$ we obtain

$$m_2(x) = \frac{x^2}{2(\alpha + 1)}$$

for all $x \in \mathbb{R}_+$.

Theorem 7.3.21 Strong laws of large numbers

7.3.21.1 for polynomial hypergroups.
Let $(\mathbb{Z}_+, *_{(Q_n)})$ be a polynomial hypergroup satisfying the assumptions of 7.3.20.1, and let $\alpha > \gamma$.

Then for any random walk $(S_n)_{n \geqslant 0}$ in $\mathbb{Z}_+$ with law $\mu \in M^1(\mathbb{Z}_+)$ such that $\mathsf{E}_*(\mu) < \infty$ we have

$$\frac{1}{n} S_n \to \mathsf{E}_*(\mu) \qquad [\mathsf{P}]\,.$$

7.3.21.2 for Sturm–Liouville hypergroups.
Let $(\mathbb{R}_+, *_A)$ be a Sturm–Liouville hypergroup with index $\varrho > 0$.

Then for any random walk $(S_n)_{n \geqslant 0}$ in $\mathbb{R}_+$ with law $\mu \in M^1(\mathbb{R}_+)$ such that $\mathsf{E}_*(\mu) < \infty$ we have

$$\frac{1}{n} S_n \to \mathsf{E}_*(\mu) \qquad [\mathsf{P}]\,.$$

For both classes of hypergroups the strong laws of large numbers remain valid if $\alpha = \gamma$ and $\varrho = 0$ respectively under additional conditions involving the speed of convergence to 0 of $(a_n - c_n)_{n \geqslant 1}$ and $\frac{A'(x)}{A(x)}$ whenever $n \to \infty$ and $x \to \infty$ respectively.

Theorem 7.3.22 Central limit theorems

7.3.22.1 for polynomial hypergroups.
Convergence to the Rayleigh distribution; polynomial growth:

Let $(\mathbb{Z}_+, *_{\alpha,\beta})$ be a Jacobi polynomial hypergroup. Then for any random walk $(S_n)_{n \geqslant 0}$ in $\mathbb{Z}_+$ with law $\mu \in M^1(\mathbb{Z}_+)$ satisfying

$$0 < \sigma^2 := \mathsf{V}_*(\mu) < \infty$$

we have

$$\mathsf{P}_{\frac{1}{\sigma\sqrt{n}}S_n} \xrightarrow{\tau_{\mathrm{w}}} \gamma_{\frac{1}{2}}^{(\alpha)},$$

where the limit measure $\gamma_{\frac{1}{2}}^{(\alpha)}$ is the Rayleigh distribution introduced in 7.2.21.

We note that the above statement says that the standardized random variables with values in a Jacobi polynomial hypergroup converge in distribution to a Gaussian measure on the Bessel–Kingman hypergroup.

Convergence to the Rayleigh distribution; exponential growth:
Let $(\mathbb{Z}_+, *_{(Q_n)})$ be a polynomial hypergroup satisfying the assumptions of 7.3.20.1, and let $\alpha > \gamma$. Moreover, suppose that

$$\sum_{n \geqslant 1} n \max(0, b_n - b_{n+1}) < \infty.$$

Let $(S_n)_{n \geqslant 0}$ be a random walk with law $\mu \neq \varepsilon_0 \in M^1(\mathbb{Z}_+)$ satisfying $\mathsf{E}_*(\mu) < \infty$ and $0 < \sigma^2 := \mathsf{V}_*(\mu) < \infty$. Then

$$\mathsf{P}_{\frac{1}{\sigma\sqrt{n}}(m_1 \circ S_n - n\,\mathsf{E}_*(\mu))} \xrightarrow{\tau_{\mathrm{w}}} N(0, 1).$$

7.3.22.2 for Sturm–Liouville hypergroups.
Convergence to the Rayleigh distribution; subexponential growth, i.e. $\varrho = 0$:
Suppose that

$$\beta := \lim_{x \to \infty} x \frac{A'(x)}{A(x)}$$

exists. Then for any random walk $(S_n)_{n \geqslant 0}$ in $\mathbb{R}_+$ with law $\mu \in M^1(\mathbb{R}_+)$ satisfying

$$0 < \sigma^2 := \mathsf{V}_*(\mu) < \infty$$

we have

$$\mathsf{P}_{\frac{1}{\sigma\sqrt{n}}S_n} \xrightarrow{\tau_{\mathrm{w}}} \gamma_{\frac{1}{2}}^{(\alpha)},$$

where $\alpha := \frac{\beta-1}{2}$.

Convergence to the Rayleigh distribution; exponential growth, i.e. $\varrho > 0$:
For any random walk $(S_n)_{n \geqslant 0}$ in $\mathbb{R}_+$ with law $\mu \in M^1(\mathbb{R}_+)$ satisfying

$$0 < \sigma^2 := \mathsf{V}_*(\mu) < \infty$$

we have

$$\mathsf{P}_{\frac{1}{\sqrt{n}}(S_n - m_1^{-1}(n\,\mathsf{E}_*(\mu)))} \xrightarrow{\tau_{\mathrm{w}}} N(0, \sigma^2).$$

In contrast to the central limit theorem cited in 7.3.22.2 for outer standardized randomized sums of the form

$$\frac{1}{f(n)} \sum_{j=1}^{n}{}^{\Lambda} X_j$$

with some real-valued function $f > 0$ on $\mathbb{N}$, one may also look at such theorems for inner standardized randomized sums like

$$\sum_{j=1}^{n} \frac{1}{f(n)} X_j \, .$$

A typical result reads as follows:

Let $(X_n)_{n \geqslant 0}$ be a sequence of independent identically distributed $\mathbb{R}_+$-valued random variables satisfying

$$\mathsf{E}(X_1^2) < \infty \, .$$

Then for the sequence $(Y_n)_{n \geqslant 1}$ of $\mathbb{R}_+$-valued random variables

$$Y_n := \sum_{j=1}^{n} \frac{1}{\sqrt{n}} X_j \qquad (n \geqslant 1)$$

we have

$$\mathsf{P}_{Y_n} \xrightarrow{\tau_{\mathrm{w}}} \gamma_{\frac{t_0}{2}}^{(\alpha)} \, ,$$

where

$$t_0 := \frac{1}{\alpha_0 + 1} \, \mathsf{E}(X_1^2)$$

with α_0 as in the Sturm–Liouville setup 7.2.7.

7.4 Increment processes and convolution semigroups

Until now we have only dealt with random walks in a commutative hypergroup. In order to obtain a wider insight into the theory of probability, more general processes with independent increments will be discussed. Such processes are related to convolution hemigroups of probability measures on the given hypergroup and similar to the classical theory they admit càdlàg paths which means that their paths are almost surely right continuous and have left hand limits. As the central topic for our exposition we choose the Lévy-type characterization of increment processes in terms of martingale properties. For Sturm–Liouville hypergroups the approach yields surprising properties of Gaussian processes.

I Modification of increment processes

Let K be a commutative hypergroup.

Definition 7.4.1 A two-parameter family $(\mu_{s,t})_{0 \leqslant s \leqslant t}$ of measures in $M^1(K)$ is called a (continuous) **convolution hemigroup** on K if

$$\mu_{s,r} * \mu_{r,t} = \mu_{s,t} \qquad \text{for all } 0 \leqslant s \leqslant r \leqslant t,$$
$$\mu_{t,t} = \varepsilon_e \qquad \text{for all } t \geqslant 0,$$

and if the mapping

$$(s,t) \mapsto \mu_{s,t}$$

from $\mathbb{S} := \{(u,v) \in \mathbb{R}^2 : 0 \leqslant u \leqslant v\}$ into $M^1(K)$ is continuous, where $M^1(K)$ as usual carries the weak topology τ_{w}.

In the following we shall need the fact that for any convolution hemigroup $(\mu_{s,t})_{0 \leqslant s \leqslant t}$ on K

$$\widehat{\mu_{s,t}}(\chi) \neq 0$$

whenever $\chi \in K^{\wedge}$ and $(s,t) \in \mathbb{S}$.

Examples 7.4.2 of convolution hemigroups.

7.4.2.1 Poisson hemigroups.
Let

$$t \mapsto \nu_t$$

be a continuous mapping from $\mathbb{R}_+$ into $M_+^{\mathrm{b}}(K)$ which is increasing in the sense that

$$\nu_t - \nu_s \in M_+^{\mathrm{b}}(K)$$

for all $0 \leqslant s \leqslant t$. Then the Poisson measures

$$\mu_{s,t} := \mathrm{e}(\nu_t - \nu_s) \qquad (0 \leqslant s \leqslant t)$$

in $M^1(K)$ form a **compound Poisson hemigroup** $(\mu_{s,t})_{0 \leqslant s \leqslant t}$ associated with the family $(\nu_t)_{t \geqslant 0}$ of measures in $M_+^{\mathrm{b}}(K)$.

7.4.2.2 Convolution semigroups.
Let $(\nu_t)_{t \geqslant 0}$ be a convolution semigroup on K. Then the measures

$$\mu_{s,t} := \nu_{t-s} \qquad (0 \leqslant s \leqslant t)$$

form a convolution hemigroup $(\mu_{s,t})_{0 \leqslant s \leqslant t}$ associated with the given convolution semigroup $(\nu_t)_{t \geqslant 0}$.

Definition 7.4.3 A family $(X_t)_{t \geqslant 0}$ of K-valued random variables on a filtered probability space $(\Omega, \mathfrak{A}, (\mathfrak{A}_t)_{t \geqslant 0}, \mathsf{P})$ is said to be a **(nonhomogeneous) increment process** in K with associated convolution hemigroup $(\mu_{s,t})_{0 \leqslant s \leqslant t}$ on K if

(a) $(X_t)_{t \geqslant 0}$ is a Markov process adapted to the filtration $(\mathfrak{A}_t)_{t \geqslant 0}$ in the sense that

$$\mathsf{P}(X_t \in B \mid \mathfrak{A}_s) = \mathsf{P}(X_t \in B \mid X_s) \qquad [\mathsf{P}]$$

for all $0 \leqslant s \leqslant t$ and all $B \in \mathfrak{B}(K)$

and if

(b) the transition probabilities of $(X_t)_{t \geqslant 0}$ are given by

$$\mathsf{P}(X_t \in B \mid X_s = x) = (\mu_{s,t} * \varepsilon_x)(B)$$

for all $0 \leqslant s \leqslant t$, $x \in K$ and $B \in \mathfrak{B}(K)$.

An increment process $(X_t)_{t \geqslant 0}$ in K is called **homogeneous** provided its associated convolution hemigroup $(\mu_{s,t})_{0 \leqslant s \leqslant t}$ is of the form

$$\mu_{s,t} = \nu_{t-s} \qquad (0 \leqslant s \leqslant t)$$

for some convolution semigroup $(\nu_t)_{t \geqslant 0}$ on K. (See Example 7.4.2.2.)

Under the assumption that K is second countable, hence metrizable with metric δ, increment processes $(X_t)_{t \geqslant 0}$ in K are named **stochastically continuous** if

$$\delta(X_s, X_t) \xrightarrow{\ \mathsf{P}\text{-stoch}\ } 0$$

as $s \to t$ for all $t \geqslant 0$.

We also note that given an arbitrary probability measure ν and a convolution hemigroup $(\mu_{s,t})_{0 \leqslant s \leqslant t}$ on K there exists an associated stochastically continuous increment process $(X_t)_{t \geqslant 0}$ on some probability space $(\Omega, \mathfrak{A}, \mathsf{P})$ taking its values in K, with initial distribution ν, since the rigth side of the equality in (b) of 7.4.3 defines a transition function

$$(x, B) \mapsto p_{s,t}(x, B)$$

on $(K, \mathfrak{B}(K))$ satisfying the Chapman–Kolmogorov equation

$$p_{s,t}(x, B) = \int_K p_{s,r}(x, \mathrm{d}y)\, p_{r,t}(y, B)$$

which holds for all $0 \leqslant s \leqslant t$, $x \in K$ and $B \in \mathfrak{B}(K)$.

As in the classical framework but with considerably more effort one proves the

Theorem 7.4.4 of modification.
If K is a second countable commutative hypergroup, then each stochastically continuous increment process $(X_t)_{t \geqslant 0}$ in K admits a modification with càdlàg paths without fixed jumps.

From now on we keep the assumption that K denotes a second countable commutative hypergroup and restrict the discussion to stochastically continuous homogeneous increment processes $(X_t)_{t \geqslant 0}$ in K which are named **Lévy processes**.

We cite a few important

Properties 7.4.5 of Lévy processes.
7.4.5.1 *Let $(X_t)_{t \geqslant 0}$ be a Lévy process associated with a convolution semigroup $(\mu_t)_{t \geqslant 0}$ on K. Then the family $(P_t)_{t \geqslant 0}$ of convolution operators*

$$P_t : C^{\mathrm{b}}(K) \to C^{\mathrm{b}}(K) \qquad (t \geqslant 0)$$

defined by

$$P_t f := \mu^- * f$$

*for all $f \in C^{\mathrm{b}}(K)$ is a **Feller semigroup** on $C^{\mathrm{b}}(K)$ with the following defining properties:*

(1) $P_t C^0(K) \subset C^0(K)$ *for all $t \geqslant 0$.*
(2) *If $f \in C^{\mathrm{b}}(K)$ with $0 \leqslant f \leqslant 1$, then*

$$0 \leqslant P_t f \leqslant 1.$$

(3) $P_s \circ P_t = P_{s+t}$ *for all* $s, t \geqslant 0$, *and* $P_0 = $ id.

(4) *For all* $f \in C^0(K)$

$$\lim_{t \to 0} \| P_t f - f \| = 0 \,.$$

Under these assumptions $(X_t)_{t \geqslant 0}$ is called a **Feller process** in K.

From 7.4.5.1 and the Dynkin–Kinney–Ray theorem ([12]) one can deduce that

7.4.5.2 *any Lévy process in K admits a càdlàg modification, a statement which we know already (compare 7.4.4).*

7.4.5.3 A Lévy process associated with a Gaussian convolution semigroup on K is called a **Gaussian process** in K.

It can be shown that each Gaussian process in K admits a continuous modification, and that each Lévy process with a.s. continuous paths is Gaussian.

As for the first assertion one makes essential use of the metric δ for K. In fact, let $(\mu_t)_{t \geqslant 0}$ be a convolution semigroup on K. For $\varepsilon > 0$ and $x \in K$ one considers the ε-neighbourhood

$$B(x, \varepsilon) := \{ z \in K : \delta(x, z) < \varepsilon \}$$

of x, and for a compact subset C of K one introduces the function

$$t \mapsto \alpha_C^\varepsilon(t) := \sup_{0 \leqslant s \leqslant t} \sup_{x \in C} (\mu_s * \varepsilon_x)\big(\complement B(x, \varepsilon)\big)$$

on $\mathbb{R}_+$. Then

$$\lim_{t \to 0} \alpha_C^\varepsilon(t) = 0 \,,$$

and, if $(\mu_t)_{t \geqslant 0}$ is a Gaussian semigroup on K, then

$$\lim_{t \to 0} \frac{1}{t} \alpha_C^\varepsilon(t) = 0 \,,$$

whenever $\varepsilon > 0$ and $C \subset K$ is compact (compare [66]).

The proof of the second assertion follows from the locality of the generator of the convolution semigroup associated with the Lévy process having a.s. continuous paths (see [18] and 7.2.17).

II Martingale characterizations of Lévy processes

We assume the reader being familiar with the basics of martingale theory and their application to stochastic calculus ([68], [93]).

All Lévy processes to be considered in the sequel are understood as Markov processes with respect to the canonical filtration.

Together with a given convolution semigroup $(\mu_t)_{t \geqslant 0}$ on K there are two important tools available: its strongly negative definite function ψ on $K^\wedge$ (7.2.16) and its generator L with domain $D(L)$ (7.2.17).

Proposition 7.4.6 *Let* $(\mu_t)_{t \geqslant 0}$ *be a convolution semigroup on* K. *For any stochastic process* $(X_t)_{t \geqslant 0}$ *in* K *the following statements are equivalent:*

(i) $(X_t)_{t \geqslant 0}$ *is a Lévy process in* K *with associated convolution semigroup* $(\mu_t)_{t \geqslant 0}$.

(ii) *For each* $\chi \in K^\wedge$ *the process*

$$\left(\widehat{\mu_t}(\overline{\chi})^{-1} \chi \circ X_t \right)_{t \geqslant 0}$$

is a martingale in $\mathbb{C}$.

(iii) *For each* $\chi \in K^\sim$ *the process*

$$\left(\widehat{\mu_t}(\overline{\chi})^{-1} \chi \circ X_t \right)_{t \geqslant 0}$$

is a martingale in $\mathbb{C}$.

Note that the implication (iii) $\Rightarrow$ (i) follows from the injectivity of the Fourier transform restricted to $K^\sim$ (7.1.14.1).

An application of Itô's integration by parts formula yields

Proposition 7.4.7 *Let* $(\mu_t)_{t \geqslant 0}$ *be a convolution semigroup on* K *with associated* $\psi \in \mathrm{SND}(K^\wedge)$, *and let* $\chi \in K^\wedge$.

For each càdlàg process $(X_t)_{t \geqslant 0}$ *in* K *the following statements are equivalent:*

(i) $\left(\widehat{\mu_t}(\overline{\chi})^{-1} \chi \circ X_t \right)_{t \geqslant 0}$ *is a martingale in* $\mathbb{C}$.

(ii) $\left(\chi \circ X_t + \psi(\overline{\chi}) \int_0^t \chi \circ X_s \, \mathrm{d}s \right)_{t \geqslant 0}$ *is a martingale in* $\mathbb{C}$.

Let $(\mu_t)_{t \geqslant 0}$ be a convolution semigroup on K with associated strongly negative definite function ψ and generator L, and let $(X_t)_{t \geqslant 0}$ be a càdlàg process in K. For any $\chi \in K^\wedge$ and $f \in D(L)$ we introduce the processes $(Y_t^{\psi, \chi})_{t \geqslant 0}$ and $(Z_t^{L,f})_{t \geqslant 0}$ by

$$Y_t^{\psi, \chi} := \chi \circ X_t - \chi \circ X_0 + \psi(\overline{\chi}) \int_0^t \chi \circ X_s \, \mathrm{d}s$$

and

$$Z_t^{L,f} := f \circ X_t - f \circ X_0 - \int_0^t L(f) \circ X_s \, \mathrm{d}s$$

for all $t \geqslant 0$ respectively.

Theorem 7.4.8 *Under the above collected assumptions the subsequent statements are equivalent:*

(i) $(X_t)_{t \geqslant 0}$ *is a Lévy process in* K *associated with* $(\mu_t)_{t \geqslant 0}$.
(ii) $\left(\widehat{\mu_t}(\overline{\chi})^{-1} \chi \circ X_t\right)_{t \geqslant 0}$ *is a martingale in* $\mathbb{C}$ *for all* $\chi \in K^{\wedge}$.
(iii) $(Y_t^{\psi, \chi})_{t \geqslant 0}$ *is a martingale in* $\mathbb{C}$ *for all* $\chi \in K^{\wedge}$.
(iv) $(Z_t^{L, f})_{t \geqslant 0}$ *is a martingale for all* $f \in D(L) \cap C^0(K)$.

It is to be noted that the implication (i) $\Rightarrow$ (iv) is related to the general martingale problem (see [19]).

In view of more useful necessary and sufficient conditions for a càdlàg process $(X_t)_{t \geqslant 0}$ in K to be a Lévy process we go back to the moment functions m_1 and m_2 of order one or two respectively and apply them to $(X_t)_{t \geqslant 0}$ in order to obtain martingales.

Remark 7.4.9 Let (m_1, m_2) be a pair of moment functions as introduced in front of 7.3.17. For a given convolution semigroup $(\mu_t)_{t \geqslant 0}$ on K we consider the pair (F_1, F_2) of functions

$$t \mapsto F_j(t) := \int_K m_j \, d\mu_t$$

on $\mathbb{R}_+$ which by definition satisfy

$$F_j(s + t) = \sum_{i=0}^{j} \binom{j}{i} F_i(s) F_{j-i}(t)$$

for all $s, t \in \mathbb{R}_+$, $j = 1, 2$, with $F_0 := 1$, since $m_0 := 1$.

Let

$$M_2^1(K) := \{\mu \in M^1(K) : m_j \in L^1(K, \mu) \text{ for } j = 1, 2\}.$$

Now we assume that a convolution semigroup $(\mu_t)_{t \geqslant 0}$ in $M_2^1(K)$ has the property that the functions F_1, F_2 are continuous at 0.

Then

(1) $F_1, F_2 \in C(\mathbb{R}_+)$,

and

(2) there exists $c_{i,j} \in \mathbb{R}$ $(i, j = 1, 2)$ such that

$$F_j(t) = \sum_{i=1}^{j} c_{i,j} t^{j+1-i}$$

for all $t \in \mathbb{R}_+$.

(3) If L is the generator of the convolution semigroup $(\mu_t^-)_{t \geqslant 0}$, then

$$Lm_j(x) = \sum_{i=1}^{j-1} m_i(x) \binom{i}{j} c_{j-i,j-i}$$

for all $x \in K$, $j = 1, 2$.

Moreover,

(4) if in the situation of (3)

$$\lim_{t \to 0} F_j(t) = 0$$

for $j = 1, 2$, then there exist constants c_1, c_2 with

(4.1) $F_1(t) = c_1 t$,

(4.2) $F_2(t) = c_1^2 t^2 + c_2 t$ for all $t \geqslant 0$.

(4.3) $Lm_1(x) = c_1$,

(4.4) $Lm_2(x) = 2c_1 m_1(x) + c_2$ for all $x \in K$.

In analogy to 7.3.18.3 one obtains

Proposition 7.4.10 *Let $(X_t)_{t \geqslant 0}$ be a Lévy process in K associated with a convolution semigroup $(\mu_t)_{t \geqslant 0}$ of measures in $M_2^1(K)$. Let (m_1, m_2) be a pair of moment functions for K with corresponding pair (F_1, F_2), and suppose that*

$$\lim_{t \to 0} F_j(t) = 0$$

for $j = 1, 2$. Then the processes

$$\big(m_1 \circ X_t - \mathsf{E}(m_1 \circ X_t)\big)_{t \geqslant 0}$$

and

$$\Big(v\big(X_t, \mathsf{E}(m_1 \circ X_t)\big) - \mathsf{E}(m_2 \circ X_t) + \mathsf{E}(m_1 \circ X_t)^2\Big)_{t \geqslant 0}$$

are martingales.

III Gaussian processes in a Sturm–Liouville hypergroup

Now we specialize the discussion of martingale characterizations to normalized Gaussian processes on a Sturm–Liouville hypergroup $(\mathbb{R}_+, *_A)$.

We define a sequence $(v_n)_{n \geqslant 0}$ of real-valued functions v_n on $(\mathbb{R}_+^\times)^2$ recursively by

$$v_0(t, x) := \int_t^x \frac{1}{A(s)} \, ds \, ,$$

$$v_{n+1}(t, x) := \int_t^x \frac{1}{A(u)} \int_u^x v_n(s, x) \, A(s) \, ds \, du \qquad (n \geqslant 0)$$

for all $(t, x) \in (\mathbb{R}_+^\times)^2$, and the corresponding sequence $(b_n)_{n \geqslant 0}$ of **canonical moment functions** b_n of order n on $\mathbb{R}_+$ by

$$b_0(x) := 1,$$

$$b_{n+1}(x) := \int_0^x v_n(t, x)\, A(t)\, \mathrm{d}t \qquad (n \geqslant 0)$$

for all $x \in \mathbb{R}_+$.

Properties 7.4.11 of canonical moment functions.

7.4.11.1 *For each $n \geqslant 1$ the function*

$$t \mapsto v_n(t, x)$$

is increasing on $]0, x]$, whenever $x \in \mathbb{R}_+^\times$.

7.4.11.2 *For $n \geqslant 0$*

$$L_A^1 v_0 = 0$$

and

$$L_A^1 v_{n+1} = -v_n,$$

where L_A^1 denotes the Sturm–Liouville operator acting on the first variable.

7.4.11.3 *For $n \geqslant 0$*

$$L_A^2 v_{n+1} = -v_n,$$

where L_A^2 denotes the Sturm–Liouville operator acting on the second variable.

From this property follows

7.4.11.4 *The canonical moments b_{n+1} are $\geqslant 0$ and increasing on $\mathbb{R}_+$ for all $n \geqslant 0$. Moreover,*

$$b_{n+1}(0) = b'_{n+1}(0) = 0$$

and

$$L_A b_{n+1} = -b_n$$

for all $n \geqslant 0$.

As a conclusion of this property we note that

7.4.11.5 *b_1, b_2 belong to the space*

$$C^{2,k} := \left\{ f \in C^2(\mathbb{R}_+) : \mathrm{Res}_{[0,\varepsilon[}\, f \in C^k([0, \varepsilon[)\ \text{for some}\ \varepsilon > 0 \right\},$$

for $k := \infty$ *and satisfy*

$$b_1(x) = c_1 x^2 + c_2 x^4 + o(x^4),$$
$$b_2(x) = c_3 x^4 + o(x^4)$$

with $c_1, c_2, c_3 > 0$, *as* $x \to 0$.

Applying a Taylor-type expansion due to Trimèche ([84], [85]) we obtain

7.4.11.6 *For all* $x, y \in \mathbb{R}_+$ *and* $n \geqslant 0$

$$\varepsilon_x * \varepsilon_y(b_n) = \sum_{k=0}^{n} b_k(x) \, b_{n-k}(y).$$

In particular, the pair $(b_1, 2b_2)$ *is a pair of moment functions for* $(\mathbb{R}_+, *_A)$.

The final answer to the comparison of the two types of moment functions is contained in property

7.4.11.7 *For any pair* (m_1, m_2) *of moment functions there exist constants* $d_1, d_2 \in \mathbb{R}$ *such that*

$$(m_1, m_2) = (d_2 b_1, d_1 b_1 + 2d_2^2 b_2).$$

Moreover, given a pair (b_1, b_2) *of canonical moment functions and constants* $d_1, d_2 \in \mathbb{R}$ *the pair*

$$(d_2 b_1, d_1 b_1 + 2d_2^2 b_2)$$

is a pair of moment functions.

7.4.11.8 *Let* $(\mu_t)_{t \geqslant 0}$ *be a Gaussian convolution semigroup on* $(\mathbb{R}_+, *_A)$. *Then for any pair* (m_1, m_2) *of moment functions and corresponding pair* (F_1, F_2) *defined in 7.4.9* $(\mu_t)_{t \geqslant 0}$ *lies in* $M_2^1(\mathbb{R}_+)$ *and satisfies*

$$\lim_{t \to 0} F_j(t) = 0$$

for $j = 1, 2$.

From 7.2.21 we know that the operator $-L_A$ is the generator of a normalized Gaussian semigroup $(\gamma_t)_{t \geqslant 0}$ on $\mathbb{R}_+$. Any continuous modification of the associated Lévy process is called a **normalized Gaussian process** in $\mathbb{R}_+$.

A direct consequence of the martingale characterization in 7.4.8 is the following

Theorem 7.4.12 *Let* $(\mathbb{R}_+, *_A)$ *be a Sturm–Liouville hypergroup. For each process* $(X_t)_{t \geqslant 0}$ *in* $\mathbb{R}_+$ *with a.s. continuous paths the statements below are equivalent:*

(i) $(X_t)_{t \geqslant 0}$ *is a Gaussian process in* $\mathbb{R}_+$.

(ii) *The process*

$$\left(\mathrm{e}^{(\lambda_\chi^2 + \varrho^2)t} \chi \circ X_t \right)_{t \geqslant 0}$$

   is a martingale for all $\chi \in \mathbb{R}_+^\sim$.

(iii) *The process*

$$\left(\chi \circ X_t + (\lambda_\chi^2 + \varrho^2) \int_0^t \chi \circ X_s \, \mathrm{d}s \right)_{t \geqslant 0}$$

   is a local martingale for all $\chi \in \mathbb{R}_+^\sim$.

(iv) $(Z_t^{-L,\chi})_{t \geqslant 0}$ *is a martingale for all* $\chi \in \mathbb{R}_+^\sim$.

(v) $(Z_t^{-L,f})_{t \geqslant 0}$ *is a martingale for all* $f \in C^{\infty,c}(\mathbb{R}_+) := C^\infty(\mathbb{R}_+) \cap C^c(\mathbb{R}_+)$ *with* $f'(0) = 0$.

(vi) $(Z_t^{-L,f})_{t \geqslant 0}$ *is a local martingale for all* $f \in C^2(\mathbb{R}_+)$ *with* $f'(0) = 0$.

In preparing the martingale characterization of normalized Gaussian processes we state

Proposition 7.4.13 *Let* $(X_t)_{t \geqslant 0}$ *be a process with a.s. continuous paths in a Sturm–Liouville hypergroup* $(\mathbb{R}_+, *_A)$, *and let* (b_1, b_2) *be a pair of canonical moment functions for* $(\mathbb{R}_+, *_A)$ *such that the processes*

$$(Z_t^{-L,b_j})_{t \geqslant 0} \qquad (j = 1, 2)$$

are local martingales. Then

(i) *the quadratic variation process of* $(Z_t^{-L,b_j})_{t \geqslant 0}$ *is a modification of the process*

$$\left(2 \int_0^t (b_1' \circ X_s)^2 \, \mathrm{d}s \right)_{t \geqslant 0} .$$

(ii) $(Z_t^{-L,f})_{t \geqslant 0}$ *is a local martingale for every* $f \in C^{2,4}(\mathbb{R}_+)$ *with* $f'(0) = f'''(0) = 0$.

The proof of this result makes strong use of Itô's stochastic calculus (see e.g. [93]).

Now we arrive at the main

Theorem 7.4.14 *Let* $(\mathbb{R}_+, *_A)$ *be a Sturm–Liouville hypergroup. For any process* $(X_t)_{t \geqslant 0}$ *in* $\mathbb{R}_+$ *with a.s. continuous paths the following statements are equivalent:*

(i) $(X_t)_{t \geqslant 0}$ *is a normalized Gaussian process in* $\mathbb{R}_+$.

(ii) *The processes*

$$(b_1 \circ X_t - t)_{t \geqslant 0}$$

  and

$$\left(b_2 \circ X_t - t\, b_1 \circ X_t + \frac{t^2}{2}\right)_{t \geqslant 0}$$

  are martingales (or local martingales).

(iii) *The process* $(Z_t^{-L,b_1})_{t \geqslant 0}$ *is a local martingale with quadratic variation process*

$$\left(\int_0^t (b_1' \circ X_s)^2 \, \mathrm{d}s\right)_{t \geqslant 0}.$$

Special case. For the Bessel–Kingman hypergroup $(\mathbb{R}_+, *_\alpha)$ with $\alpha \geqslant -\frac{1}{2}$ the canonical moments are given by

$$b_1(x) = \frac{x^2}{4(\alpha + 1)}$$

and

$$b_2(x) = \frac{x^4}{32(\alpha + 1)(\alpha + 2)}$$

for all $x \in \mathbb{R}_+$ respectively.

From Theorem 7.4.14 one deduces immediately a Lévy-type martingale characterization.

Theorem 7.4.15 *For any process* $(X_t)_{t \geqslant 0}$ *in* $\mathbb{R}_+$ *with a.s. continuous paths the statements below are equivalent:*

(i) $(X_t)_{t \geqslant 0}$ *is a Bessel process of order* $\alpha \geqslant -\frac{1}{2}$.

(ii) *The processes*

$$(X_t^2 - 2(\alpha + 1)t)_{t \geqslant 0}$$

  and

$$\left(X_t^4 - 4(\alpha + 2)tX_t^2 + 4(\alpha + 1)(\alpha + 2)t^2\right)_{t \geqslant 0}$$

  are martingales (or local martingales).

Remark 7.4.16 Most of the results in this section concerning homogeneous (Lévy) increment processes have been extended to nonhomogeneous increment processes (see [45]).

Comments on the selection of references

The main source for the material exposed in the present chapter is the monograph [10]. For historical reasons we added the surveys [38] and [40] which indicate the growth of the theory during the last two decades. The layout of Sections 7.3 and 7.4 has been influenced by the papers [25] and [66] respectively. In Section 7.4 we arrive at part of the actual research: Lévy-type martingale characterizations for convolution semigroups and hemigroups on groups have been studied in [90] and [44] respectively. For hypergroups the problem has been investigated in [66] and [45] respectively.

As general references to the theory of stochastic processes we mention the books [4] for basic knowledge, [12], [18] for Markov processes and [19], [68], [93] for semimartingales and Itô calculus. The references [100], [11], [65] contain first extensions of one-variable polynomial and one-dimensional Sturm–Liouville hypergroups to higher dimensions. More general product hypergroups are discussed in [26]. The papers [71], [92] and the lecture notes [73] relate hypergroups to the framework of Dunkl operators. There are only a few citations in the text which refer exclusively to technical details.

Chapter 8

Limit Theorems on Locally Compact Abelian Groups

by Gyula Pap

Limit theorems will be proved for row sums of a rowwise independent infinitesimal array of random variables with values in a locally compact Abelian group. First we give a proof of Gaiser's theorem (Satz 1.3.6 in [24]), since it does not have an easy access and it is not complete. This theorem gives sufficient conditions for convergence in distribution of the row sums, but the limit measure cannot have a nondegenerate idempotent factor. Then necessary and sufficient conditions will be proved for convergence in distribution of the row sums, where the limit measure can also be a nondegenerate Haar measure on a compact subgroup. Finally, special cases will be investigated: the torus group, the group of p-adic integers and the p-adic solenoid.

8.1 Limit problems and parametrization of weakly infinitely divisible measures

Let G be a second countable locally compact Abelian T_0-topological group. The main question of limit problems on G can be formulated as follows. Let

$$(X_{n,k})_{n \in \mathbb{N}; \, k=1,\ldots,K_n}$$

be a *triangular array* of rowwise independent random variables with values in G satisfying the *infinitesimality* condition

$$\lim_{n \to \infty} \max_{1 \leqslant k \leqslant K_n} \mathsf{P}(X_{n,k} \in \complement U) = 0$$

for all Borel neighbourhoods $U \in \mathfrak{V}(0)$. Let $\mathrm{LM}(G)$ denote the set of all possible limits of row sums of rowwise independent infinitesimal triangular arrays in G. The following problems arise:

(P1) How to parametrize the set $\mathrm{LM}(G)$, i.e. to give a bijection between $\mathrm{LM}(G)$ and an appropriate parameter set $\mathcal{P}(G)$;

(P2) How to associate suitable quantities q_n, $n \in \mathbb{N}$ to the rows

$$(X_{n,k})_{k=1,\ldots,K_n}, \qquad n \in \mathbb{N}$$

so that

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_{\mathrm{w}}} \mu \quad \Longleftrightarrow \quad q_n \to q,$$

where $q \in \mathcal{P}(G)$ corresponds to the limit distribution μ, and the convergence $q_n \to q$ is meant in an appropriate sense.

The problem (P1) has been solved by Parthasarathy (see Chapter IV, Corollary 7.1 in [60] and Remark 8.1.4). It turns out that any measure $\mu \in \mathrm{LM}(G)$ is necessarily weakly infinitely divisible. Gaiser [24] gave a partial solution to the problem (P2). His theorem (see Theorem 8.2.6) gives only some sufficient conditions for the convergence

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_{\mathrm{w}}} \mu,$$

which does not include the case where μ has a nondegenerate idempotent factor, i.e. a nondegenerate Haar measure on a compact subgroup of G. For a survey of results on limit theorems on a general locally compact Abelian topological group see Bingham [9].

In this chapter necessary and sufficient conditions will be proved for some limit theorems to hold on general locally compact Abelian groups. These results complete the results of the paper [24]. In these theorems the limit measure can also be a nondegenerate Haar measure on a compact subgroup of G.

These results will also be specified considering some classical groups such as the torus group, the group of p-adic integers and the p-adic solenoid. Here Gaiser's theorem will also be applied. For completeness, we present a proof of this theorem, since Gaiser's dissertation does not have an easy access and Gaiser's proof is not complete. Concerning limit problems on totally disconnected Abelian topological groups, like the group of p-adic integers, we mention Telöken [82].

Definition 8.1.1 A measure $\mu \in M^1(G)$ is called **weakly infinitely divisible** if for all $n \in \mathbb{N}$ there exist a measure $\mu_n \in M^1(G)$ and an element $x_n \in G$ such that $\mu = \mu_n^n * \varepsilon_{x_n}$. The collection of all weakly infinitely divisible measures on G will be denoted by $\mathrm{WI}(G)$.

According to Corollary 7.1 in Chapter IV of Parthasarathy [60], $\mathrm{LM}(G) \subset \mathrm{WI}(G)$. The aim of the following discussion is to parametrize the set $\mathrm{WI}(G)$ by describing the Fourier transform of measures in $\mathrm{WI}(G)$.

First we recall the building blocks of weakly infinitely divisible measures. If K is a compact subgroup of G and ω_K denotes the normed Haar measure of K in the sense that $\omega_K \in M^1(G)$ then by 4.4.20 its Fourier transform has the form

$$\widehat{\omega_K}(\chi) = \begin{cases} 1 & \text{if } \chi(x) = 1 \text{ for all } x \in K, \\ 0 & \text{otherwise,} \end{cases} \qquad \chi \in G^{\wedge}, \qquad (1)$$

that is

$$\widehat{\omega_K} = \mathbb{1}_{K^{\perp}}.$$

Clearly $\omega_K \in \mathrm{WI}(G)$, since $\omega_K * \omega_K = \omega_K$. Sazonov and Tutubalin [77] proved that $\omega_K \in \mathrm{LM}(G)$.

Obviously $\varepsilon_x \in \mathrm{WI}(G)$ for all $x \in G$, and one can easily check that $\varepsilon_x \in \mathrm{LM}(G)$ for all $x \in G^{\mathrm{arc}}$, where G^{arc} denotes the arc-component of the identity 0.

For any nonnegative continuous quadratic form ψ on $G^{\wedge}$ there exists a unique measure $\gamma_{\psi} \in M^1(G)$ determined by

$$\widehat{\gamma_{\psi}}(\chi) = \mathrm{e}^{-\psi(\chi)/2} \qquad \text{for all } \chi \in G^{\wedge},$$

which is a symmetric Gaussian measure (see Theorem 5.4.15). Obviously $\gamma_{\psi} \in \mathrm{LM}(G)$ since $\gamma_{\psi} = \gamma_{\psi/n}^n$ for all $n \in \mathbb{N}$ and $\tau_{\mathrm{w}}\text{–}\lim_{n \to \infty} \gamma_{\psi/n} = \varepsilon_0$.

Definition 8.1.2 For a Lévy measure $\eta \in \mathrm{L}(G)$ and for a Lévy function g for G the **compound Poisson measure** $\pi_{\eta, g} \in M^1(G)$ is defined by

$$\widehat{\pi_{\eta, g}}(\chi) = \exp\left\{ \int_{G^{\times}} \big(\chi(x) - 1 - \mathrm{i}g(x, \chi)\big)\, \eta(\mathrm{d}x) \right\} \qquad \text{for all } \chi \in G^{\wedge}$$

(see Theorem 5.4.15).

Obviously $\pi_{\eta, g} \in \mathrm{LM}(G)$ since $\pi_{\eta, g} = \pi_{\eta/n, g}^n$ for all $n \in \mathbb{N}$ and $\tau_{\mathrm{w}}\text{–}\lim_{n \to \infty} \pi_{\eta/n, g} = \varepsilon_0$.

Let $\mathcal{P}(G)$ be the set of all quadruplets (K, a, ψ, η), where K is a compact subgroup of G, $a \in G$, ψ is a nonnegative continuous quadratic form on $G^{\wedge}$ and $\eta \in \mathrm{L}(G)$ is a Lévy measure. The following parametrization of weakly infinitely divisible measures on G is known (see Parthasarathy [60], Chapter IV, Corollary 7.1).

Theorem 8.1.3 (Canonical representation of weakly infinitely divisible measures)
Let g be a Lévy function for G. If $\mu \in \mathrm{WI}(G)$ then there exists a quadruplet $(K, a, \psi, \eta) \in \mathcal{P}(G)$ such that

$$\mu = \omega_K * \varepsilon_a * \gamma_\psi * \pi_{\eta, g}. \tag{2}$$

Conversely, if $(K, a, \psi, \eta) \in \mathcal{P}(G)$ then $\omega_K * \varepsilon_a * \gamma_\psi * \pi_{\eta, g} \in \mathrm{WI}(G)$.

Remark 8.1.4 In general, this parametrization is not one-to-one (see page 112, Remark 3 of Parthasarathy [60]), but the compact subgroup K is uniquely determined in (2) by μ (more precisely, K is the annihilator of the open subgroup $\{\chi \in G^\wedge : \widehat{\mu}(\chi) \neq 0\}$). If $K = \{0\}$ then the quadratic form ψ in (2) is also uniquely determined by μ. In order to obtain one-to-one parametrization one can take equivalence classes of quadruplets related to the equivalence relation $\approx$ defined by

$$(K, a_1, \psi_1, \eta_1) \approx (K, a_2, \psi_2, \eta_2)$$
$$\iff \quad \omega_H * \varepsilon_{a_1} * \gamma_{\psi_1} * \pi_{\eta_1, g} = \omega_H * \varepsilon_{a_2} * \gamma_{\psi_2} * \pi_{\eta_2, g}.$$

8.2 Gaiser's limit theorem

We need the notions of local mean of bounded measures and of special Lévy functions.

Definition 8.2.1 For a bounded measure η on G and for a Lévy function g for G, the **local mean** of η with respect to g is the uniquely defined element $m_g(\eta) \in G$ given by

$$\chi(m_g(\eta)) = \exp\left\{ i \int_G g(x, \chi) \, \eta(\mathrm{d}x) \right\} \qquad \text{for all } \chi \in G^\wedge.$$

The existence and uniqueness of a local mean is guaranteed by Pontryagin's duality theorem 4.2.13. If η coincides with the distribution P_X of a random variable X in G, we will use the notation $m_g(X)$ instead of $m_g(\mathsf{P}_X)$. Remark that $\chi(m_g(X)) = e^{i\,\mathsf{E}(g(X,\chi))}$ for all $\chi \in G^\wedge$. Moreover, for a measure $\eta \in M^{\mathrm{b}}(G)$ with $\eta(\{0\}) = 0$ we have $\eta \in \mathrm{L}(G)$ and $\mathrm{e}(\eta) = \pi_{\eta,\,g} * \varepsilon_{m_g(\eta)}$.

Definition 8.2.2 A Lévy function g for G is called **special** if it is uniformly continuous in its first variable, i.e. if for all $\varepsilon > 0$ there exists $U \in \mathfrak{V}(0)$ such that $|g(x,\chi) - g(y,\chi)| < \varepsilon$ for all $x, y \in G$ with $x - y \in U$.

Gaiser proved the existence of a special Lévy function for an arbitrary second countable locally compact Abelian T_0-topological group (see Satz 1.1.4 in [24]). The proof goes along the lines of the proof of Theorem 5.3.4.

We also need the following version of the well-known Portemanteau theorem for not necessarily bounded measures. Let $C_0(G)$ and $C_0^{\mathrm{u}}(G)$ denote the set of all functions in $C^{\mathrm{b}}(G)$ vanishing in some $U \in \mathfrak{V}(0)$, and the set of all uniformly continuous functions in $C_0(G)$ respectively.

Theorem 8.2.3 *Let $(\eta_n)_{n\in\mathbb{Z}_+}$ be a sequence of extended real-valued measures on G such that $\eta_n(\complement U) < \infty$ for all $U \in \mathfrak{V}(0)$ and for all $n \in \mathbb{Z}_+$. Then the following assertions are equivalent:*

(1) $\displaystyle\int_G f \, \mathrm{d}\eta_n \to \int_G f \, \mathrm{d}\eta_0$ *as* $n \to \infty$ *for all* $f \in C_0(G)$,

(2) $\displaystyle\int_G f \, \mathrm{d}\eta_n \to \int_G f \, \mathrm{d}\eta_0$ *as* $n \to \infty$ *for all* $f \in C_0^{\mathrm{u}}(G)$,

(3) $\eta_n(\complement U) \to \eta_0(\complement U)$ *as* $n \to \infty$ *for all* $U \in \mathfrak{V}(0)$ *with* $\eta_0(\partial U) = 0$,

(4) $\displaystyle\int_{\complement U} f \, d\eta_n \to \int_{\complement U} f \, d\eta_0$ *as* $n \to \infty$

   for all $f \in C^{\mathrm{b}}(G)$, $U \in \mathfrak{V}(0)$ *with* $\eta_0(\partial U) = 0$,

(5) $\eta_n|_{\complement U} \xrightarrow{\tau_w} \eta_0|_{\complement U}$ *as* $n \to \infty$ *for all* $U \in \mathfrak{V}(0)$ *with* $\eta_0(\partial U) = 0$.

For the equivalence of (a) and (c) see Meerschaert and Scheffler [58], Proposition 1.2.19, for the rest see Barczy and Pap [2].

First we recall a theorem about convergence of weakly infinitely divisible measures without idempotent factors (see Satz 1.2.1 of Gaiser [24]).

Theorem 8.2.4 *For each* $n \in \mathbb{Z}_+$, *let* $\mu_n \in \mathrm{WI}(G)$ *be such that* (2) *holds for* μ_n *with a quadruplet* $(\{0\}, a_n, \psi_n, \eta_n)$. *If there exists a Lévy function* g *for* G *such that*

(i) $a_n \to a_0$ *as* $n \to \infty$,

(ii) $\displaystyle\psi_n(\chi) + \int_{G^\times} g(x, \chi)^2 \, \eta_n(dx) \to \psi_0(\chi) + \int_{G^\times} g(x, \chi)^2 \, \eta_0(dx)$ $\forall \chi \in G^\wedge$,

(iii) $\displaystyle\int_{G^\times} f \, d\eta_n \to \int_{G^\times} f \, d\eta_0$ *as* $n \to \infty$ *for all* $f \in C_0(G)$,

then

$$\mu_n \xrightarrow{\tau_w} \mu_0 \qquad as \;\; n \to \infty.$$

Proof. It suffices to show $\widehat{\mu_n}(\chi) \to \widehat{\mu_0}(\chi)$ as $n \to \infty$ for all $\chi \in G^\wedge$. Let

$$h(x, \chi) := \chi(x) - 1 - ig(x, \chi) + \frac{1}{2}g(x, \chi)^2$$

for all $x \in G$ and all $\chi \in G^\wedge$. Then

$$\widehat{\mu_n}(\chi) = \chi(a_n) \exp\left\{ -\frac{1}{2}\left(\psi_n(\chi) + \int_{G^\times} g(x, \chi)^2 \, \eta_n(dx) \right) \right.$$
$$\left. + \int_{G^\times} h(x, \chi) \, \eta_n(dx) \right\}$$

for all $n \in \mathbb{Z}_+$ and all $\chi \in G^\wedge$. Taking into account assumptions (i) and (ii), it is enough to show that

$$\int_{G^\times} h(x, \chi) \, \eta_n(dx) \to \int_{G^\times} h(x, \chi) \, \eta_0(dx) \quad as \;\; n \to \infty \qquad (3)$$

for all $\chi \in G^\wedge$. For each $\chi \in G^\wedge$ there exists $U \in \mathfrak{V}(0)$ such that

$$\chi(x) = e^{ig(x, \chi)} \qquad \text{for all } x \in U.$$

Using the inequality

$$\left| e^{iy} - 1 - iy + \frac{y^2}{2} \right| \leqslant \frac{|y|^3}{6} \qquad \text{for all } y \in \mathbb{R}, \tag{4}$$

we obtain

$$|h(x, \chi)| \leqslant \frac{|g(x, \chi)|^3}{6} \qquad \text{for all } x \in U.$$

Consequently, for all $V \in \mathfrak{V}(0)$ with $V \subset U$

$$\left| \int_{G^\times} h(x, \chi)\, \eta_n(\mathrm{d}x) - \int_{G^\times} h(x, \chi)\, \eta_0(\mathrm{d}x) \right| \leqslant I_n^{(1)}(V) + I_n^{(2)}(V),$$

where

$$I_n^{(1)}(V) := \frac{1}{6} \int_V |g(x, \chi)|^3\, (\eta_n + \eta_0)(\mathrm{d}x),$$

$$I_n^{(2)}(V) := \left| \int_{\complement V} h(x, \chi)\, \eta_n(\mathrm{d}x) - \int_{\complement V} h(x, \chi)\, \eta_0(\mathrm{d}x) \right|.$$

We have

$$I_n^{(1)}(V) \leqslant \frac{1}{6} \sup_{x \in V} |g(x, \chi)| \int_V g(x, \chi)^2\, (\eta_n + \eta_0)(\mathrm{d}x).$$

By assumption (ii),

$$\sup_{n \in \mathbb{Z}_+} \int_V g(x, \chi)^2\, \eta_n(\mathrm{d}x) \leqslant \sup_{n \in \mathbb{Z}_+} \left(\psi_n(\chi) + \int_{G^\times} g(x, \chi)^2\, \eta_n(\mathrm{d}x) \right) < \infty.$$

Theorem 8.3 in Hewitt and Ross [36] yields existence of a metric d on G compatible with the topology of G. The function

$$t \mapsto \eta_0(\{x \in G : d(x, 0) \geqslant t\})$$

from $(0, \infty)$ into $\mathbb{R}$ is non-increasing, hence the set

$$\{t \in (0, \infty) : \eta_0(\{x \in G : d(x, 0) = t\}) > 0\}$$

of its discontinuities is countable. Consequently, for arbitrary $\varepsilon > 0$, there exists $t > 0$ such that

$$V_1 := \{x \in G : d(x, 0) < t\} \in \mathfrak{V}(0), \qquad V_1 \subset U, \qquad \eta_0(\partial V_1) = 0$$

and

$$\sup_{y \in V_1} |g(x, \chi)| < \frac{3\varepsilon}{2 \sup_{n \in \mathbb{Z}_+} \int_V g(x, \chi)^2\, \eta_n(\mathrm{d}x)},$$

thus $I_n^{(1)}(V_1) < \varepsilon/2$. By assumption (iii) and Theorem 8.2.3, $I_n^{(2)}(V_1) < \varepsilon/2$ for all sufficiently large n, hence we obtain

$$\left| \int_{G^\times} h(x, \chi)\, \eta_n(\mathrm{d}x) - \int_{G^\times} h(x, \chi)\, \eta_0(\mathrm{d}x) \right| < \varepsilon$$

for all sufficiently large n, which implies (3) for all $\chi \in G^\wedge$. $\qquad\square$

We shall refer to the following theorem about the accompanying Poisson array due to Parthasarathy (see Chapter IV, Theorem 5.1 in [60]).

Theorem 8.2.5   *Let* g *be a fixed Lévy function for* G. *Let* $(X_{n,k})_{n\in\mathbb{N};\, k=1,\dots,K_n}$ *be a rowwise independent infinitesimal array of random variables in* G. *For each* $n \in \mathbb{N}$, *let*

$$\mu_n := \mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}}, \qquad \nu_n := \mathrm{e}\left(\sum_{k=1}^{K_n} \mathsf{P}_{X_{n,k}-m_g(X_{n,k})}\right) * \varepsilon_{\sum_{k=1}^{K_n} m_g(X_{n,k})}.$$

If any of the sequences $(\mu_n)_{n\in\mathbb{N}}$ *and* $(\nu_n)_{n\in\mathbb{N}}$ *is shift compact and no limit of its shifts has an idempotent factor then*

$$\lim_{n\to\infty} \sup_{\chi\in C} |\widehat{\mu_n}(\chi) - \widehat{\nu_n}(\chi)| = 0$$

for each compact set C *of* $G^\wedge$.

Gaiser proved the following limit theorem (see Satz 1.3.6 in [24]).

Theorem 8.2.6   *Let* g *be a fixed Lévy function for* G. *Let* $(X_{n,k})_{n\in\mathbb{N};\, k=1,\dots,K_n}$ *be a rowwise independent infinitesimal array of random variables in* G. *Suppose that there exists a quadruplet* $(\{0\}, a, \psi, \eta) \in \mathcal{P}(G)$ *such that*

(i) $\displaystyle\sum_{k=1}^{K_n} m_g(X_{n,k}) \to a$ *as* $n \to \infty$,

(ii) $\displaystyle\sum_{k=1}^{K_n} \mathsf{V}(g(X_{n,k}, \chi)) \to \psi(\chi) + \int_{G^\times} g(x, \chi)^2\, \eta(\mathrm{d}x)$ *for all* $\chi \in G^\wedge$,

(iii) $\displaystyle\sum_{k=1}^{K_n} \mathsf{E}(f(X_{n,k})) \to \int_{G^\times} f\, \mathrm{d}\eta$ *as* $n \to \infty$ *for all* $f \in C_0(G)$.

Then

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\ \tau_{\mathrm{w}}\ } \varepsilon_a * \gamma_\psi * \pi_{\eta,\, g} \qquad \text{as}\ \ n \to \infty. \tag{5}$$

Proof.   First we show that it is enough to prove the statement for a special Lévy function, namely, if the statement is true for some Lévy function g, then it is true for any Lévy function $\widetilde{g}$. Suppose that assumptions (i)–(iii) hold for $\widetilde{g}$ with a quadruplet $(\{0\}, a, \psi, \eta)$. We show that they hold for g with the quadruplet

$$(\{0\},\, a + m_{g,\widetilde{g}}(\eta),\, \psi,\, \eta)\,,$$

where the element $m_{g,\widetilde{g}}(\eta) \in G$ is uniquely determined by

$$\chi(m_{g,\widetilde{g}}(\eta)) = \exp\left\{i\int_{G^\times} (g(x,\chi) - \widetilde{g}(x,\chi))\, \eta(\mathrm{d}x)\right\} \qquad \text{for all } \chi \in G^\wedge.$$

(Note that $g(\cdot, \chi) - \widetilde{g}(\cdot, \chi) \in C_0(G)$ can be checked easily.) Hence we want to prove

(i') $\displaystyle\sum_{k=1}^{K_n} m_g(X_{n,k}) \to a + m_{g,\widetilde{g}}(\eta)$ as $n \to \infty$,

(ii') $\displaystyle\sum_{k=1}^{K_n} \mathsf{V}(g(X_{n,k}, \chi)) \to \psi(\chi) + \int_{G^\times} g(x,\chi)^2\, \eta(\mathrm{d}x)$ for all $\chi \in G^\wedge$,

(iii') $\displaystyle\sum_{k=1}^{K_n} \mathsf{E}(f(X_{n,k})) \to \int_{G^\times} f\, \mathrm{d}\eta$ as $n \to \infty$ for all $f \in C_0(G)$.

Clearly (iii') holds, since it is identical with assumption (iii).
By assumption (i), in order to prove (i') we have to show

$$\chi\left(\sum_{k=1}^{K_n} m_g(X_{n,k}) - \sum_{k=1}^{K_n} m_{\widetilde{g}}(X_{n,k})\right) \to \chi(m_{g,\widetilde{g}}(\eta)) \quad \text{for all } \chi \in G^\wedge.$$

We have

$$\chi\left(\sum_{k=1}^{K_n} m_g(X_{n,k}) - \sum_{k=1}^{K_n} m_{\widetilde{g}}(X_{n,k})\right) = \prod_{k=1}^{K_n} \frac{\chi(m_g(X_{n,k}))}{\chi(m_{\widetilde{g}}(X_{n,k}))}$$

$$= \prod_{k=1}^{K_n} \frac{e^{i\,\mathsf{E}(g(X_{n,k},\chi))}}{e^{i\,\mathsf{E}(\widetilde{g}(X_{n,k},\chi))}} = \exp\left\{i\sum_{k=1}^{K_n} \mathsf{E}\left(g(X_{n,k},\chi) - \widetilde{g}(X_{n,k},\chi)\right)\right\}$$

$$\to \exp\left\{i\int_{G^\times} (g(x,\chi) - \widetilde{g}(x,\chi))\, \eta(\mathrm{d}x)\right\},$$

where we applied assumption (iii) with the function $g(\cdot,\chi) - \widetilde{g}(\cdot,\chi) \in C_0(G)$.
By assumption (ii), in order to prove (ii') we have to show

$$\sum_{k=1}^{K_n} \mathsf{V}(g(X_{n,k},\chi)) - \sum_{k=1}^{K_n} \mathsf{V}(\widetilde{g}(X_{n,k},\chi))$$

$$\to \int_{G^\times} \left(g(x,\chi)^2 - \widetilde{g}(x,\chi)^2\right) \eta(\mathrm{d}x) \tag{6}$$

for all $\chi \in G^\wedge$, where $g(\cdot, \chi)^2 - \widetilde{g}(\cdot, \chi)^2 \in C_0(G)$ can be checked easily. We have

$$\sum_{k=1}^{K_n} \mathsf{V}(g(X_{n,k}, \chi)) - \sum_{k=1}^{K_n} \mathsf{V}(\widetilde{g}(X_{n,k}, \chi)) = A_n - B_n,$$

where

$$A_n := \sum_{k=1}^{K_n} \mathsf{E}\left(g(X_{n,k}, \chi)^2 - \widetilde{g}(X_{n,k}, \chi)^2\right),$$

$$B_n := \sum_{k=1}^{K_n} \left[(\mathsf{E}(g(X_{n,k}, \chi)))^2 - (\mathsf{E}(\widetilde{g}(X_{n,k}, \chi)))^2\right].$$

Applying assumption (iii) with the function $g(\cdot, \chi)^2 - \widetilde{g}(\cdot, \chi)^2 \in C_0(G)$, we obtain

$$A_n \to \int_{G^\times} \left(g(x, \chi)^2 - \widetilde{g}(x, \chi)^2\right) \eta(\mathrm{d}x). \tag{7}$$

Moreover,

$$B_n = \sum_{k=1}^{K_n} \mathsf{E}\left(g(X_{n,k}, \chi) - \widetilde{g}(X_{n,k}, \chi)\right) \mathsf{E}\left(g(X_{n,k}, \chi) + \widetilde{g}(X_{n,k}, \chi)\right)$$

implies

$$|B_n| \leqslant \max_{1 \leqslant k \leqslant K_n} \mathsf{E}\left(|g(X_{n,k}, \chi)| + |\widetilde{g}(X_{n,k}, \chi)|\right)$$

$$\times \sum_{k=1}^{K_n} \mathsf{E}\left|g(X_{n,k}, \chi) - \widetilde{g}(X_{n,k}, \chi)\right|.$$

Using assumption (iii) with the function $|g(\cdot, \chi) - \widetilde{g}(\cdot, \chi)| \in C_0(G)$, we get

$$\sum_{k=1}^{K_n} \mathsf{E}\left|g(X_{n,k}, \chi) - \widetilde{g}(X_{n,k}, \chi)\right| \to \int_{G^\times} |g(x, \chi) - \widetilde{g}(x, \chi)| \eta(\mathrm{d}x). \tag{8}$$

Infinitesimality of $(X_{n,k})_{n \in \mathbb{N}; k=1,\ldots,K_n}$ implies

$$\max_{1 \leqslant k \leqslant K_n} \mathsf{E}\left|g(X_{n,k}, \chi)\right| \to 0 \qquad \text{for all } \chi \in G^\wedge. \tag{9}$$

Indeed,

$$\max_{1 \leqslant k \leqslant K_n} \mathsf{E}\left|g(X_{n,k}, \chi)\right|$$

$$\leqslant \sup_{x \in U} |g(x, \chi)| + \sup_{x \in G} |g(x, \chi)| \cdot \max_{1 \leqslant k \leqslant K_n} \mathsf{P}(X_{n,k} \in \complement U)$$

for all $U \in \mathfrak{V}(0)$ and for all $\chi \in G^\wedge$, and (LF4) of Definition 5.3.1 implies $\sup_{x \in U} |g(x, \chi)| \to 0$ as $U \to \{0\}$. Clearly (8) and (9) imply $B_n \to 0$, hence, by (7), we obtain (6).

We conclude that assumptions (i)–(iii) hold for g with the quadruplet $(\{0\}, a + m_{g, \tilde{g}}(\eta), \psi, \eta)$. Since we supposed that the statement is true for g, we get

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_{\mathrm{w}}} \varepsilon_{a + m_{g, \tilde{g}}(\eta)} * \gamma_\psi * \pi_{\eta, g} .$$

Hence

$$\mathsf{E}\chi\left(\sum_{k=1}^{K_n} X_{n,k}\right)$$

$$\to \chi(a + m_{g, \tilde{g}}(\eta)) \exp\left\{-\frac{1}{2}\psi(\chi) + \int_{G^\times} \left(\chi(x) - 1 - ig(x, \chi)\right) \eta(\mathrm{d}x)\right\}$$

$$= \chi(a) \exp\left\{-\frac{1}{2}\psi(\chi) + \int_{G^\times} \left(\chi(x) - 1 - i\tilde{g}(x, \chi)\right) \eta(\mathrm{d}x)\right\}$$

for all $\chi \in G^\wedge$, which implies

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_{\mathrm{w}}} \varepsilon_a * \gamma_\psi * \pi_{\eta, \tilde{g}} .$$

Thus we may suppose that g is a special Lévy function. Let

$$Y_{n,k} := X_{n,k} - m_g(X_{n,k}) \qquad \text{for all } n \in \mathbb{N}, \ k = 1, \ldots, K_n.$$

We show that $(Y_{n,k})_{n \in \mathbb{N}; k=1,\ldots,K_n}$ is an infinitesimal array of rowwise independent random variables in G, and

(i'') $\displaystyle\sum_{k=1}^{K_n} m_g(Y_{n,k}) \to 0$ as $n \to \infty$,

(ii'') $\displaystyle\sum_{k=1}^{K_n} \mathsf{E}\left(g(Y_{n,k}, \chi)^2\right) \to \psi(\chi) + \int_{G^\times} g(x, \chi)^2 \, \eta(\mathrm{d}x)$ for all $\chi \in G^\wedge$,

(iii'') $\displaystyle\sum_{k=1}^{K_n} \mathsf{E}(f(Y_{n,k})) \to \int_{G^\times} f \, \mathrm{d}\eta$ as $n \to \infty$ for all $f \in C_0(G)$.

Infinitesimality of $(Y_{n,k})_{n \in \mathbb{N}; k=1,\ldots,K_n}$ is equivalent to

$$\max_{1 \leqslant k \leqslant K_n} |\mathsf{E}(\chi(Y_{n,k})) - 1| \to 0 \qquad \text{for all } \chi \in G^\wedge. \tag{10}$$

We have
$$\left| \mathsf{E}(\chi(Y_{n,k})) - 1 \right| = \left| \frac{\mathsf{E}(\chi(X_{n,k}))}{\chi(m_g(X_{n,k}))} - 1 \right| = \left| \frac{\mathsf{E}(\chi(X_{n,k}))}{\mathrm{e}^{\mathrm{i}\,\mathsf{E}\,g(X_{n,k},\chi)}} - 1 \right|$$

$$= \left| \mathsf{E}(\chi(X_{n,k})) - \mathrm{e}^{\mathrm{i}\,\mathsf{E}(g(X_{n,k},\chi))} \right|$$

$$\leqslant \left| \mathsf{E}(\chi(X_{n,k})) - 1 \right| + \left| \mathrm{e}^{\mathrm{i}\,\mathsf{E}(g(X_{n,k},\chi))} - 1 \right|.$$

Infinitesimality of $(X_{n,k})_{n\in\mathbb{N};\,k=1,\dots,K_n}$ implies
$$\max_{1\leqslant k\leqslant K_n} \left| \mathsf{E}(\chi(X_{n,k})) - 1 \right| \to 0 \qquad \text{for all} \ \ \chi \in G^\wedge. \tag{11}$$

Infinitesimality of $(X_{n,k})_{n\in\mathbb{N};\,k=1,\dots,K_n}$ implies (9) as well, hence using the inequality $|\mathrm{e}^{\mathrm{i}y} - 1| \leqslant |y|$ for all $y \in \mathbb{R}$, we get
$$\max_{1\leqslant k\leqslant K_n} \left| \mathrm{e}^{\mathrm{i}\,\mathsf{E}(g(X_{n,k},\chi))} - 1 \right| \to 0 \qquad \text{for all} \ \ \chi \in G^\wedge,$$
and we obtain (10).

For (i″), it is enough to show
$$\sum_{k=1}^{K_n} \mathsf{E}(g(Y_{n,k},\chi)) \to 0 \qquad \text{for all} \ \ \chi \in G^\wedge.$$

Let $\chi \in G^\wedge$ be fixed. Infinitesimality of $(X_{n,k})_{n\in\mathbb{N};\,k=1,\dots,K_n}$ implies that for all $V \in \mathfrak{V}(0)$ and for all sufficiently large n we have
$$m_g(X_{n,k}) \in V \qquad \text{for} \ \ k = 1, \dots, K_n. \tag{12}$$
Consequently, using (9) as well, we conclude that for all sufficiently large n we have
$$g(m_g(X_{n,k}),\chi) = \mathsf{E}(g(X_{n,k},\chi)) \qquad \text{for} \ \ k = 1, \dots, K_n. \tag{13}$$
Infinitesimality of $(X_{n,k})_{n\in\mathbb{N};\,k=1,\dots,K_n}$ and properties (LF3) and (LF4) of the Lévy function g imply also the existence of $U \in \mathfrak{V}(0)$ such that
$$g(x - m_g(X_{n,k}),\chi) = g(x,\chi) - g(m_g(X_{n,k}),\chi)$$
$$\text{for all} \ \ x \in U, \ \ k = 1, \dots, K_n \tag{14}$$
for all sufficiently large n (see Parthasarathy [60], page 91). Consequently, we obtain
$$\left| \sum_{k=1}^{K_n} \mathsf{E}(g(Y_{n,k},\chi)) \right|$$

$$= \left| \sum_{k=1}^{K_n} \mathsf{E}\left(g(Y_{n,k},\chi) - g(X_{n,k},\chi) + g(m_g(X_{n,k}),\chi) \right) \mathbb{1}_{\complement U}(X_{n,k}) \right|$$

$$\leqslant \left(\max_{1\leqslant k\leqslant K_n} \sup_{x\in G} |g(x - m_g(X_{n,k}),\chi) - g(x,\chi)| \right) \sum_{k=1}^{K_n} \mathsf{P}(X_{n,k} \in \complement U)$$

$$+ \max_{1\leqslant k\leqslant K_n} |g(m_g(X_{n,k}),\chi)| \sum_{k=1}^{K_n} \mathsf{P}(X_{n,k} \in \complement U) \to 0$$

for all sufficiently large n. Indeed, by (12),

$$\max_{1 \leqslant k \leqslant K_n} \sup_{x \in G} |g(x - m_g(X_{n,k}), \chi) - g(x, \chi)| \to 0 \qquad \text{as } n \to \infty, \qquad (15)$$

since g is uniformly continuous in its first variable being a *special* Lévy function.

Moreover, (9) and (13) imply

$$\max_{1 \leqslant k \leqslant K_n} |g(m_g(X_{n,k}), \chi)| \to 0 \qquad \text{as } n \to \infty, \qquad (16)$$

and assumption (iii) implies

$$\sup_{n \in \mathbb{N}} \sum_{k=1}^{K_n} \mathsf{P}(X_{n,k} \in \complement U) < \infty. \qquad (17)$$

To prove (ii″), we have to show

$$\sum_{k=1}^{K_n} \Big(\mathsf{E}\big(g(Y_{n,k}, \chi)^2\big) - \mathsf{V}(g(X_{n,k}, \chi)) \Big) \to 0 \qquad \text{for all } \chi \in G^\wedge.$$

Consider again a neighbourhood $U \in \mathfrak{V}(0)$ such that (14) holds for all sufficiently large n. We have

$$\mathsf{E}\big(g(Y_{n,k}, \chi)^2\big) - \mathsf{V}(g(X_{n,k}, \chi)) = C_{n,k} + D_{n,k},$$

where

$$C_{n,k} := \mathsf{E}\Big(g(Y_{n,k}, \chi)^2 - g(X_{n,k}, \chi)^2\Big)\mathbb{1}_U(X_{n,k}) + \big(\mathsf{E}\, g(X_{n,k}, \chi)\big)^2,$$

$$D_{n,k} := \mathsf{E}\Big(g(Y_{n,k}, \chi)^2 - g(X_{n,k}, \chi)^2\Big)\mathbb{1}_{\complement U}(X_{n,k}).$$

For all sufficiently large n we have (13), hence

$$
\begin{aligned}
C_{n,k} &= \mathsf{E}\Big(\big(g(X_{n,k}, \chi) - g(m_g(X_{n,k}), \chi)\big)^2 - g(X_{n,k}, \chi)^2\Big)\mathbb{1}_U(X_{n,k}) \\
&\quad + \big(\mathsf{E}(g(X_{n,k}, \chi))\big)^2 \\
&= g(m_g(X_{n,k}), \chi)^2\, \mathsf{P}(X_{n,k} \in U) + \big(\mathsf{E}(g(X_{n,k}, \chi))\big)^2 \\
&\quad - 2g(m_g(X_{n,k}), \chi)\, \mathsf{E}\big(g(X_{n,k}, \chi)\mathbb{1}_U(X_{n,k})\big) \\
&= 2\,\mathsf{E}(g(X_{n,k}, \chi))\, \mathsf{E}\big(g(X_{n,k}, \chi)\mathbb{1}_{\complement U}(X_{n,k})\big) \\
&\quad - \big(\mathsf{E}(g(X_{n,k}, \chi))\big)^2\, \mathsf{P}(X_{n,k} \in \complement U).
\end{aligned}
$$

Consequently, again by (13),

$$|C_{n,k}| \leqslant \mathsf{P}(X_{n,k} \in \complement U)$$

$$\times \left(2|\mathsf{E}(g(X_{n,k}, \chi))|\sup_{x \in G}|g(x, \chi)| + |\mathsf{E}(g(X_{n,k}, \chi))|^2 \right). \qquad (18)$$

Moreover,

$$D_{n,k} = \mathsf{E}\big(g(Y_{n,k},\chi) - g(X_{n,k},\chi)\big)\big(g(Y_{n,k},\chi) + g(X_{n,k},\chi)\big)\mathbb{1}_{\complement U}(X_{n,k}),$$

thus

$$
\begin{aligned}
|D_{n,k}| &\leqslant 2\,\mathsf{P}(X_{n,k} \in \complement U)\sup_{x \in G}|g(x,\chi)| \\
&\times \max_{1 \leqslant k \leqslant K_n} \sup_{x \in G}\big|g(x - m_g(X_{n,k}),\chi) - g(x,\chi)\big|.
\end{aligned}
\tag{19}
$$

Now (18) and (19), using (15), (16) and (17), imply (ii'').
   To prove (iii''), it is enough to show

$$\sum_{k=1}^{K_n} \mathsf{E}(f(Y_{n,k})) - \sum_{k=1}^{K_n} \mathsf{E}(f(X_{n,k})) \to 0 \tag{20}$$

for all $f \in C_0^{\mathrm{u}}(G)$ (see Theorem 8.2.3). Choose $V \in \mathfrak{V}(0)$ such that $f(x) = 0$ for all $x \in V$. Then choose $U \in \mathfrak{V}(0)$ such that $U - U \subset V$, where $U - U := \{x - y : x, y \in U\}$. Infinitesimality of $(X_{n,k})_{n\in\mathbb{N};\,k=1,\dots,K_n}$ implies that for all sufficiently large n we have $m_g(X_{n,k}) \in U$ for $k = 1, \dots, K_n$, hence $f(Y_{n,k}) - f(X_{n,k}) = \big(f(Y_{n,k}) - f(X_{n,k})\big)\mathbb{1}_{\complement U}(X_{n,k})$. Consequently,

$$
\left|\sum_{k=1}^{K_n} \mathsf{E}(f(Y_{n,k})) - \sum_{k=1}^{K_n} \mathsf{E}(f(X_{n,k}))\right|
$$

$$
\leqslant \sup_{x \in G}\big|f(x - m_g(X_{n,k})) - f(x)\big| \sum_{k=1}^{K_n} \mathsf{P}(X_{n,k} \in \complement U),
$$

and uniform continuity of f and (17) imply (20).
   Now consider the shifted compound Poisson measures

$$\nu_n := \mathrm{e}\left(\sum_{k=1}^{K_n}\mathsf{P}_{Y_{n,k}}\right) * \varepsilon_{\sum_{k=1}^{K_n} m_g(X_{n,k})}, \qquad n \in \mathbb{N}.$$

Clearly $\nu_n \in \mathrm{WI}(G)$ such that (2) holds for ν_n with the quadruplet

$$\left(\{0\}, \sum_{k=1}^{K_n} m_g(X_{n,k}) + \sum_{k=1}^{K_n} m_g(Y_{n,k}), 0, \sum_{k=1}^{K_n}\mathsf{P}_{Y_{n,k}}\right).$$

By Theorem 8.2.4, using (i) and (i'')–(iii''), we obtain

$$\nu_n \xrightarrow{\tau_{\mathrm{w}}} \varepsilon_a * \gamma_\psi * \pi_{\eta,\,g}.$$

Applying Theorem 8.2.5 we conclude the statement. $\square$

Remark 8.2.7 If either $a \neq 0$ or $\psi \neq 0$ or $\eta \neq 0$ then the infinitesimality of $(X_{n,k})_{n\in\mathbb{N};\,k=1,\dots,K_n}$ and (5) imply $K_n \to \infty$.

Remark 8.2.8 Condition (i) of Theorem 8.2.6 is equivalent to

(i′) $\exp\left\{ i \sum_{k=1}^{K_n} \mathsf{E}(g(X_{n,k},\chi)) \right\} \to \chi(a)$ as $n \to \infty$ for all $\chi \in G^{\wedge}$.

Remark 8.2.9 Due to Theorem 8.2.3, condition (iii) of Theorem 8.2.6 is equivalent to

(iii′) $\sum_{k=1}^{K_n} \mathsf{P}(X_{n,k} \in \complement U) \to \eta(\complement U)$ as $n \to \infty$

for all $U \in \mathfrak{V}(0)$ with $\eta(\partial U) = 0$.

8.3 Limit theorems for symmetric arrays and Bernoulli arrays

First we derive some limit theorems for symmetric arrays. The following result is an easy consequence of Theorem 8.2.6.

Theorem 8.3.1 (CLT for symmetric arrays)
Let g be a fixed Lévy function for G. Let $(X_{n,k})_{n\in\mathbb{N};\, k=1,\ldots,K_n}$ be a rowwise independent array of random variables in G such that $X_{n,k} \stackrel{d}{=} -X_{n,k}$ for all $n \in \mathbb{N}$, $k = 1, \ldots, K_n$. Suppose that there exists a quadratic form ψ on $G^\wedge$ such that

(i) $\displaystyle\sum_{k=1}^{K_n} V(g(X_{n,k}, \chi)) \to \psi(\chi)$ *as $n \to \infty$ for all $\chi \in G^\wedge$,*

(ii) $\displaystyle\sum_{k=1}^{K_n} P(X_{n,k} \in \complement U) \to 0$ *as $n \to \infty$ for all $U \in \mathfrak{V}(0)$.*

Then the array $(X_{n,k})_{n\in\mathbb{N};\, k=1,\ldots,K_n}$ is infinitesimal and

$$P_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_w} \gamma_\psi \qquad as \ \ n \to \infty.$$

For the proof of Theorem 8.3.3, we need the following simple observation.

Lemma 8.3.2 *Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\alpha_n \geqslant -n$ for all sufficiently large n, and let $\alpha \in \mathbb{R} \cup \{-\infty, \infty\}$. Then*

$$\left(1 + \frac{\alpha_n}{n}\right)^n \to e^\alpha \qquad \Longleftrightarrow \qquad \alpha_n \to \alpha,$$

where $e^{-\infty} := 0$ and $e^\infty := \infty$.

Proof. If $\alpha_n \to \alpha \in \mathbb{R}$ then $\alpha_n/n \to 0$, hence l'Hospital's rule gives

$$\log\left[\left(1 + \frac{\alpha_n}{n}\right)^n\right] = \alpha_n \cdot \frac{\log(1 + \alpha_n/n)}{\alpha_n/n} \to \alpha.$$

If $\alpha_n \to -\infty$ then we choose $n_0 \in \mathbb{N}$ such that $\alpha_n \geqslant -n$ for all $n \geqslant n_0$, hence $1 + \alpha_n/n \geqslant 0$ for all $n \geqslant n_0$, implying $\liminf_{n\to\infty}(1 + \alpha_n/n)^n \geqslant 0$. For each $M \in \mathbb{R}$ there exists $n_M \in \mathbb{N}$ such that $\alpha_n \leqslant M$ for all $n \geqslant n_M$. Then $(1 + \alpha_n/n)^n \leqslant (1 + M/n)^n$ for all $n \geqslant \max(n_0, n_M)$, which implies

$$\limsup_{n\to\infty}\left(1 + \frac{\alpha_n}{n}\right)^n \leqslant \limsup_{n\to\infty}\left(1 + \frac{M}{n}\right)^n = e^M.$$

Since M is arbitrary, we obtain $\limsup_{n\to\infty}(1 + \alpha_n/n)^n \leqslant 0$, and finally $\lim_{n\to\infty}(1 + \alpha_n/n)^n = 0$. The case of $\alpha_n \to \infty$ can be handled similarly.

If $(1 + \alpha_n/n)^n \to e^\alpha$ and $\alpha_n \not\to \alpha$ then there exist subsequences (n') and (n'') and $\alpha', \alpha'' \in \mathbb{R} \cup \{-\infty, \infty\}$ with $\alpha' \neq \alpha''$ such that $\alpha_{n'} \to \alpha'$ and $\alpha_{n''} \to \alpha''$. Then $(1 + \alpha_{n'}/n')^{n'} \to e^{\alpha'}$ and $(1 + \alpha_{n''}/n'')^{n''} \to e^{\alpha''}$ lead to a contradiction. $\qquad\square$

The next theorem gives necessary and sufficient conditions in case of a rowwise independent and identically distributed (i.i.d.) symmetric array. It turns out that in this special case conditions of Theorem 8.3.1 are not only sufficient but necessary as well. If G is compact then the limit measure can be the normalized Haar measure on G.

Theorem 8.3.3 (Limit theorem for rowwise i.i.d. symmetric arrays)
Let $(X_{n,k})_{n\in\mathbb{N};\, k=1,\ldots,K_n}$ be an infinitesimal, rowwise i.i.d. array of random variables in G such that $K_n \to \infty$ and $X_{n,k} \stackrel{d}{=} -X_{n,k}$ for all $n \in \mathbb{N}$, $k = 1, \ldots, K_n$.

If g is a Lévy function for G and ψ is a quadratic form on $G^\wedge$, then the following statements are equivalent:

(i) $\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_w} \gamma_\psi$ *as* $n \to \infty$,

(ii) $K_n\big(1 - \operatorname{Re}\mathsf{E}(\chi(X_{n,1}))\big) \to \dfrac{\psi(\chi)}{2}$ *as* $n \to \infty$ *for all* $\chi \in G^\wedge$,

(iii) $K_n\,\mathsf{V}(g(X_{n,1}, \chi)) \to \psi(\chi)$ *as* $n \to \infty$ *for all* $\chi \in G^\wedge$ *and*
$\quad\ K_n\,\mathsf{P}(X_{n,1} \in \complement U) \to 0$ *as* $n \to \infty$ *for all* $U \in \mathfrak{V}(0)$.

If G is compact then the following statements are equivalent:

(iv) $\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_w} \omega_G$

(v) $K_n\big(1 - \operatorname{Re}\mathsf{E}(\chi(X_{n,1}))\big) \to \infty$ *for all* $\chi \in G^\wedge \setminus \{\mathbb{1}_G\}$.

Proof. (i) $\Longleftrightarrow$ (ii). Statement (i) is equivalent to

$$\mathsf{E}\chi\bigg(\sum_{k=1}^{K_n} X_{n,k}\bigg) \to \widehat{\gamma}_\psi(\chi) \qquad \text{for all } \chi \in G^\wedge. \tag{21}$$

We have $\widehat{\gamma}_\psi(\chi) = \mathrm{e}^{-\psi(\chi)/2}$. Clearly $X_{n,k} \overset{\mathrm{d}}{=} -X_{n,k}$ implies $\mathsf{E}(\chi(X_{n,k})) = \mathrm{Re}\,\mathsf{E}(\chi(X_{n,k}))$, hence

$$\mathsf{E}\,\chi\left(\sum_{k=1}^{K_n} X_{n,k}\right) = \big(\mathrm{Re}\,\mathsf{E}(\chi(X_{n,1}))\big)^{K_n}$$

$$= \left(1 + \frac{K_n\big(\mathrm{Re}\,\mathsf{E}(\chi(X_{n,1})) - 1\big)}{K_n}\right)^{K_n}. \tag{22}$$

Infinitesimality of $(X_{n,k})_{n\in\mathbb{N};\,k=1,\dots,K_n}$ implies $\mathsf{E}(\chi(X_{n,1})) \to 1$ (see (10)), thus $\mathrm{Re}\,\mathsf{E}\,\chi(X_{n,1}) - 1 \geqslant -1$ for all sufficiently large $n \in \mathbb{N}$. Hence by $K_n \to \infty$ and by Lemma 8.3.2 we conclude that (21) and (ii) are equivalent.

(ii) $\implies$ (iii). We have already proved that (ii) implies (i), hence, by Theorem 5.4.2 in Heyer [37], (ii) implies $K_n\,\mathsf{P}(X_{n,1} \in \complement U) \to 0$ for all $U \in \mathfrak{V}(0)$. Clearly $X_{n,k} \overset{\mathrm{d}}{=} -X_{n,k}$ implies $\mathsf{E}(g(X_{n,k},\chi)) = 0$, thus $\mathsf{V}(g(X_{n,1},\chi)) = \mathsf{E}\big(g(X_{n,1},\chi)^2\big)$. Consequently, it is enough to show

$$K_n\left(\mathrm{Re}\,\mathsf{E}(\chi(X_{n,1})) - 1 + \frac{1}{2}\,\mathsf{E}\big(g(X_{n,1},\chi)^2\big)\right) \to 0 \qquad \forall\,\chi \in G^\wedge. \tag{23}$$

For $\chi \in G^\wedge$, choose $U \in \mathfrak{V}(0)$ such that $\chi(x) = \mathrm{e}^{ig(x,\chi)}$ and

$$\frac{1}{4}g(x,\chi)^2 \leqslant 1 - \mathrm{Re}\,\chi(x) \leqslant \frac{1}{2}g(x,\chi)^2 \qquad \text{for all } x \in U. \tag{24}$$

Then

$$K_n\left(\mathrm{Re}\,\mathsf{E}(\chi(X_{n,1})) - 1 + \frac{1}{2}\,\mathsf{E}\big(g(X_{n,1},\chi)^2\big)\right) = A_n + B_n\,,$$

where

$$A_n := K_n\,\mathrm{Re}\,\mathsf{E}\left(\mathrm{e}^{ig(X_{n,1},\chi)} - 1 - ig(X_{n,1},\chi) + \frac{1}{2}g(X_{n,1},\chi)^2\right)\mathbb{1}_U(X_{n,1})\,,$$

$$B_n := K_n\,\mathrm{Re}\,\mathsf{E}\left(\chi(X_{n,1}) - 1 + \frac{1}{2}g(X_{n,1},\chi)^2\right)\mathbb{1}_{\complement U}(X_{n,1})\,.$$

By formulas (4) and (24) we get

$$|A_n| \leqslant \frac{1}{6}K_n\,\mathsf{E}\big(|g(X_{n,1},\chi)|^3\,\mathbb{1}_U(X_{n,1})\big) \leqslant \frac{4\big(K_n(1 - \mathrm{Re}\,\mathsf{E}(\chi(X_{n,1})))\big)^{3/2}}{3K_n^{1/2}}\,,$$

hence $K_n \to \infty$ and assumption (ii) yield $A_n \to 0$. Moreover,

$$|B_n| \leqslant \left(2 + \frac{1}{2}\sup_{x\in G} g(x,\chi)^2\right)K_n\,\mathsf{P}(X_{n,1} \in \complement U) \to 0\,,$$

thus we obtain (23).

(iii) $\Longrightarrow$ (i) follows from Theorem 8.3.1.

(iv) $\Longleftrightarrow$ (v). Convergence

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_w} \omega_G$$

is equivalent to

$$\mathsf{E}\,\chi\Big(\sum_{k=1}^{K_n} X_{n,k} \Big) \to \widehat{\omega}_G(\chi) \qquad \text{for all } \chi \in G^\wedge. \tag{25}$$

Using (22), (1) and Lemma 8.3.2, one can easily show that (25) holds if and only if $K_n\big(1 - \operatorname{Re}\mathsf{E}(\chi(X_{n,1}))\big) \to \infty$ for all $\chi \in G^\wedge \setminus \{\mathbb{1}_G\}$. $\qquad\square$

The next statement is a special case of Theorem 8.3.3 for *Rademacher arrays*.

Theorem 8.3.4 (Limit theorem for rowwise i.i.d. Rademacher arrays)
Let $x_n \in G$, $n \in \mathbb{N}$ such that $x_n \to 0$. Let $(X_{n,k})_{n\in\mathbb{N};\,k=1,\ldots,K_n}$ be a rowwise i.i.d. array of random variables in G such that $K_n \to \infty$ and

$$\mathsf{P}(X_{n,k} = x_n) = \mathsf{P}(X_{n,k} = -x_n) = \frac{1}{2}\,.$$

Then the array $(X_{n,k})_{n\in\mathbb{N};\,k=1,\ldots,K_n}$ is infinitesimal.

If ψ is a quadratic form on $G^\wedge$ then the following statements are equivalent:

(i) $\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_w} \gamma_\psi$ *as $n \to \infty$,*

(ii) $K_n\big(1 - \operatorname{Re}\chi(x_n)\big) \to \dfrac{\psi(\chi)}{2}$ *as $n \to \infty$ for all $\chi \in G^\wedge$.*

If G is compact then the following statements are equivalent:

(iii) $\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_w} \omega_G$ *as $n \to \infty$,*

(iv) $K_n\big(1 - \operatorname{Re}\chi(x_n)\big) \to \infty$ *as $n \to \infty$ for all $\chi \in G^\wedge \setminus \{\mathbb{1}_G\}$.*

Note that in Theorem 8.3.4 the expression $1 - \operatorname{Re}\chi(x_n)$ can be replaced in both places by $\frac{1}{2}g(x_n, \chi)^2$, where g is an arbitrary Lévy function for G (see the proof of (23) and the inequalities in (24)).

Next we derive some limit theorems for *Bernoulli arrays*. In the following limit theorem the limit measure can be the normalized Haar measure on an arbitrary compact subgroup of G generated by a single element.

Theorem 8.3.5 (Limit theorem for rowwise i.i.d. Bernoulli array)
Let $x \in G$ such that $x \neq 0$. Let $(X_{n,k})_{n\in\mathbb{N};\,k=1,\ldots,K_n}$ be a rowwise i.i.d. array of random variables in G such that $K_n \to \infty$,

$$\mathsf{P}(X_{n,k} = x) = p_n\,, \qquad \mathsf{P}(X_{n,k} = 0) = 1 - p_n\,,$$

and $p_n \to 0$. *Then the array* $(X_{n,k})_{n \in \mathbb{N}; \, k=1,\ldots,K_n}$ *is infinitesimal.*

If λ *is a nonnegative real number then*

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_{\mathrm{w}}} \mathrm{e}(\lambda \varepsilon_x) \quad \Longleftrightarrow \quad K_n \, p_n \to \lambda.$$

If the smallest closed subgroup H *of* G *containing* x *is compact then*

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_{\mathrm{w}}} \omega_H \quad \Longleftrightarrow \quad K_n \, p_n \to \infty.$$

Proof. First we suppose $K_n \, p_n \to \lambda$ and show convergence

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_{\mathrm{w}}} \mathrm{e}(\lambda \varepsilon_x).$$

We need to prove

$$\mathsf{E}\chi\left(\sum_{k=1}^{K_n} X_{n,k}\right) \to (\mathrm{e}(\lambda \varepsilon_x))^{\wedge}(\chi) \qquad \text{for all } \chi \in G^{\wedge}. \tag{26}$$

We have $(\mathrm{e}(\lambda \varepsilon_x))^{\wedge}(\chi) = \mathrm{e}^{\lambda(\chi(x)-1)}$ and

$$\begin{aligned}
\mathsf{E}\chi\left(\sum_{k=1}^{K_n} X_{n,k}\right) &= (p_n \chi(x) + 1 - p_n)^{K_n} \\
&= \left(1 + \frac{K_n \, p_n(\chi(x)-1)}{K_n}\right)^{K_n}.
\end{aligned} \tag{27}$$

If $(z_n)_{n \in \mathbb{N}}$ is a sequence of complex numbers such that $z_n \to z \in \mathbb{C}$ then $\left(1 + \frac{z_n}{n}\right)^n \to \mathrm{e}^z$. Consequently, $K_n \, p_n \to \lambda$ and $K_n \to \infty$ imply (26).

Next we suppose $K_n \, p_n \to \infty$ and show that

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_{\mathrm{w}}} \omega_H.$$

We need to prove

$$\mathsf{E}\chi\left(\sum_{k=1}^{K_n} X_{n,k}\right) \to \widehat{\omega}_H(\chi) \qquad \text{for all } \chi \in G^{\wedge}.$$

According to Remarks 23.24 in Hewitt and Ross [36], $\{x\}^{\perp} = H^{\perp}$, and thus by (1) we are left to check

$$\mathsf{E}\chi\left(\sum_{k=1}^{K_n} X_{n,k}\right) \to \begin{cases} 1 & \text{if } \chi \in \{x\}^{\perp}, \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

If $\chi \in \{x\}^{\perp}$ then $\chi(x) = 1$, hence

$$\mathsf{E}\chi\left(\sum_{k=1}^{K_n} X_{n,k}\right) = (p_n \chi(x) + 1 - p_n)^{K_n} = 1,$$

and we obtain (28). To handle the case $\chi \notin \{x\}^{\perp}$, consider the equality

$$\left| \mathsf{E}\,\chi\left(\sum_{k=1}^{K_n} X_{n,k}\right)\right| = |p_n \chi(x) + 1 - p_n|^{K_n}$$

$$= \left(\left(1 + p_n(\operatorname{Re}\chi(x) - 1)\right)^2 + p_n^2\left(\operatorname{Im}\chi(x)\right)^2\right)^{K_n/2}$$

$$= \left(1 + \frac{K_n\,p_n\left(2(\operatorname{Re}\chi(x) - 1) + p_n|1 - \chi(x)|^2\right)}{K_n}\right)^{K_n/2}.$$

Clearly $\chi \notin \{x\}^{\perp}$ implies $\chi(x) \neq 1$, and by $|\chi(x)| = 1$ we get $\operatorname{Re}\chi(x) - 1 < 0$. Hence, by Lemma 8.3.2, we conclude that $K_n\,p_n \to \infty$, $K_n \to \infty$ and $p_n \to 0$ imply (28).

Now we suppose

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_w} \mathrm{e}(\lambda\varepsilon_x)$$

and derive $K_n\,p_n \to \lambda$. If $K_n\,p_n \nrightarrow \lambda$ then either there exists a subsequence (n') such that $K_{n'}\,p_{n'} \to \infty$, or there exist subsequences (n'') and (n''') and two distinct nonnegative real numbers λ'' and λ''' such that $K_{n''}\,p_{n''} \to \lambda''$ and $K_{n'''}\,p_{n'''} \to \lambda'''$. In the first case we would obtain

$$\mathsf{P}_{\sum_{k=1}^{K_{n'}} X_{n',k}} \xrightarrow{\tau_w} \omega_H,$$

which contradicts to

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_w} \mathrm{e}(\lambda\varepsilon_x).$$

In the second case we would obtain

$$\mathsf{P}_{\sum_{k=1}^{K_{n''}} X_{n'',k}} \xrightarrow{\tau_w} \mathrm{e}(\lambda''\varepsilon_x) \quad \text{and} \quad \mathsf{P}_{\sum_{k=1}^{K_{n'''}} X_{n''',k}} \xrightarrow{\tau_w} \mathrm{e}(\lambda'''\varepsilon_x)$$

which again contradicts to

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_w} \mathrm{e}(\lambda\varepsilon_x).$$

Finally we suppose

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_w} \omega_H$$

and prove $K_n\,p_n \to \infty$. If $K_n\,p_n \nrightarrow \infty$ then there exists a subsequence (n') and a nonnegative real number λ' such that $K_{n'}\,p_{n'} \to \lambda'$. Then we would obtain

$$\mathsf{P}_{\sum_{k=1}^{K_{n'}} X_{n',k}} \xrightarrow{\tau_w} \mathrm{e}(\lambda'\varepsilon_x),$$

which contradicts to

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_w} \omega_H. \qquad \square$$

8.4 Limit theorems for special locally compact Abelian groups

First we consider the 1-dimensional *torus group*. The set $\mathbb{T} := \{e^{ix} : -\pi \leqslant x < \pi\}$ equipped with the usual multiplication of complex numbers is a second countable compact Abelian T_0-topological group with identity $e := e^0 = 1$. This is called the 1-dimensional torus group. Its character group is $\mathbb{T}^\wedge = \{\chi_\ell : \ell \in \mathbb{Z}\}$, where

$$\chi_\ell(y) := y^\ell, \qquad y \in \mathbb{T}, \quad \ell \in \mathbb{Z}.$$

Hence $\mathbb{T}^\wedge$ can be identified with the additive group of integers $\mathbb{Z}$. The compact subgroups of $\mathbb{T}$ are

$$H_r := \{e^{2\pi ij/r} : j = 0, 1, \ldots, r - 1\}, \qquad r \in \mathbb{N},$$

and $\mathbb{T}$ itself.

The set of all quadratic forms on $\mathbb{T}^\wedge \cong \mathbb{Z}$ is $\{\psi_b : b \in \mathbb{R}_+\}$, where

$$\psi_b(\chi_\ell) := b\ell^2, \qquad \ell \in \mathbb{Z}, \quad b \in \mathbb{R}_+.$$

Let us define the functions $\arg : \mathbb{T} \to [-\pi, \pi[$ and $h : \mathbb{R} \to \mathbb{R}$ by

$$\arg(e^{ix}) := x, \qquad -\pi \leqslant x < \pi,$$

$$h(x) := \begin{cases} 0 & \text{if } x < -\pi \text{ or } x \geqslant \pi, \\ -x - \pi & \text{if } -\pi \leqslant x < -\pi/2, \\ x & \text{if } -\pi/2 \leqslant x < \pi/2, \\ -x + \pi & \text{if } \pi/2 \leqslant x < \pi. \end{cases}$$

An extended real-valued measure η on $\mathbb{T}$ is a Lévy measure if and only if

$$\eta(\{e\}) = 0 \quad \text{and} \quad \int_{\mathbb{T}^\times} (\arg y)^2 \, \eta(\mathrm{d}y) < \infty.$$

By Example 5.3.2.2 the function $g_{\mathbb{T}} : \mathbb{T} \times \mathbb{T}^\wedge \to \mathbb{R}$, defined by

$$g_{\mathbb{T}}(y, \chi_\ell) := \ell h(\arg y), \qquad y \in \mathbb{T}, \quad \ell \in \mathbb{Z},$$

is a Lévy function for $\mathbb{T}$.

Theorem 8.2.6 has the following consequence on the torus.

Theorem 8.4.1 (Gauss–Poisson limit theorem)
Let $(X_{n,k})_{n\in\mathbb{N};\, k=1,\ldots,K_n}$ be a rowwise independent array of random variables in $\mathbb{T}$. Suppose that there exists a quadruplet $(\{e\}, a, \psi_b, \eta) \in \mathcal{P}(\mathbb{T})$ such that

(i) $\displaystyle\max_{1\leqslant k\leqslant K_n} \mathsf{P}(|\arg(X_{n,k})| > \varepsilon) \to 0$ *as* $n \to \infty$ *for all* $\varepsilon > 0$,

(ii) $\displaystyle\exp\left\{ i\sum_{k=1}^{K_n} \mathsf{E}(h(\arg(X_{n,k}))) \right\} \to a$ *as* $n \to \infty$,

(iii) $\displaystyle\sum_{k=1}^{K_n} \mathsf{V}(h(\arg(X_{n,k}))) \to b + \int_{\mathbb{T}^\times} \left(h(\arg y) \right)^2 \eta(\mathrm{d}y)$ *as* $n \to \infty$,

(iv) $\displaystyle\sum_{k=1}^{K_n} \mathsf{E}(f(X_{n,k})) \to \int_{\mathbb{T}^\times} f \, \mathrm{d}\eta$ *as* $n \to \infty$ *for all* $f \in C_0(\mathbb{T})$.

Then the array $(X_{n,k})_{n\in\mathbb{N};\, k=1,\ldots,K_n}$ *is infinitesimal and*

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\ \tau_{\mathrm{w}}\ } \varepsilon_a * \gamma_{\psi_b} * \pi_{\eta,\, g_{\mathbb{T}}} \qquad \text{as } n \to \infty.$$

If the limit measure has no compound Poisson factor $\pi_{\eta,\, g_{\mathbb{T}}}$ then the truncation function h can be omitted.

Theorem 8.4.2 (CLT)
Let $(X_{n,k})_{n\in\mathbb{N};\, k=1,\ldots,K_n}$ *be a rowwise independent array of random variables in* $\mathbb{T}$. *Suppose that there exist an element* $a \in \mathbb{T}$ *and a nonnegative real number* b *such that*

(i) $\displaystyle\exp\left\{ i\sum_{k=1}^{K_n} \mathsf{E}(\arg(X_{n,k})) \right\} \to a$ *as* $n \to \infty$,

(ii) $\displaystyle\sum_{k=1}^{K_n} \mathsf{V}(\arg(X_{n,k})) \to b$ *as* $n \to \infty$,

(iii) $\displaystyle\sum_{k=1}^{K_n} \mathsf{P}(|\arg(X_{n,k})| > \varepsilon) \to 0$ *as* $n \to \infty$ *for all* $\varepsilon > 0$.

Then the array $(X_{n,k})_{n\in\mathbb{N};\, k=1,\ldots,K_n}$ *is infinitesimal and*

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\ \tau_{\mathrm{w}}\ } \varepsilon_a * \gamma_{\psi_b} \qquad \text{as } n \to \infty.$$

Proof.   In view of Theorem 8.4.1 and Remark 8.2.9, it is enough to check

(i′) $\displaystyle\exp\left\{ i\sum_{k=1}^{K_n} \mathsf{E}(h(\arg(X_{n,k}))) \right\} \to a$ *as* $n \to \infty$,

(ii′) $\displaystyle\sum_{k=1}^{K_n} \mathsf{V}(h(\arg(X_{n,k}))) \to b$ *as* $n \to \infty$,

(iii′) $\displaystyle\sum_{k=1}^{K_n} \mathsf{P}(|\arg(X_{n,k})| > \varepsilon) \to 0$ as $n \to \infty$ for all $\varepsilon > 0$.

Clearly (iii′) and assumption (iii) are identical. In order to prove (i′) it is sufficient to show

$$\sum_{k=1}^{K_n} \mathsf{E}(h(\arg(X_{n,k}))) - \sum_{k=1}^{K_n} \mathsf{E}(\arg(X_{n,k})) \to 0 \,,$$

since $|\mathrm{e}^{\mathrm{i}y_1} - \mathrm{e}^{\mathrm{i}y_2}| = |\mathrm{e}^{\mathrm{i}(y_1 - y_2)} - 1| \leqslant |y_1 - y_2|$ for all $y_1, y_2 \in \mathbb{R}$. We have

$$|h(y) - y| \leqslant \pi \mathbb{1}_{[-\pi, -\pi/2] \cup [\pi/2, \pi]}(y) \qquad \text{for all } y \in [-\pi, \pi],$$

hence

$$\left| \sum_{k=1}^{K_n} \mathsf{E}(h(\arg(X_{n,k}))) - \sum_{k=1}^{K_n} \mathsf{E}(\arg(X_{n,k})) \right|$$

$$\leqslant \pi \sum_{k=1}^{K_n} \mathsf{P}(|\arg(X_{n,k})| \geqslant \pi/2) \to 0$$

by condition (iii). In order to check (ii′) it is enough to prove

$$\sum_{k=1}^{K_n} \mathsf{V}(h(\arg(X_{n,k}))) - \sum_{k=1}^{K_n} \mathsf{V}(\arg(X_{n,k})) \to 0.$$

We have

$$\left| \sum_{k=1}^{K_n} \mathsf{V}(h(\arg(X_{n,k}))) - \sum_{k=1}^{K_n} \mathsf{V}(\arg(X_{n,k})) \right|$$

$$\leqslant \sum_{k=1}^{K_n} \mathsf{E} \left| \left(h(\arg(X_{n,k})) \right)^2 - (\arg(X_{n,k}))^2 \right|$$

$$+ \sum_{k=1}^{K_n} \left| \left(\mathsf{E}(h(\arg(X_{n,k}))) \right)^2 - (\mathsf{E}(\arg(X_{n,k})))^2 \right|$$

$$\leqslant 2\pi^2 \sum_{k=1}^{K_n} \mathsf{P}(|\arg(X_{n,k})| \geqslant \pi/2) \to 0 \,,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 8.3.4 has the following consequence on the torus.

Theorem 8.4.3 (Limit theorem for rowwise i.i.d. Rademacher array)
Let $x_n \in \mathbb{T}$, $n \in \mathbb{N}$ such that $x_n \to e$. Let $(X_{n,k})_{n \in \mathbb{N};\, k=1,\ldots,K_n}$ be a rowwise i.i.d. array of random variables in $\mathbb{T}$ such that $K_n \to \infty$ and

$$\mathsf{P}(X_{n,k} = x_n) = \mathsf{P}(X_{n,k} = -x_n) = \frac{1}{2}\,.$$

Then the array $(X_{n,k})_{n \in \mathbb{N};\, k=1,\ldots,K_n}$ is infinitesimal.
　　If b is a nonnegative real number then

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_\mathrm{w}} \gamma_{\psi_b} \qquad \Longleftrightarrow \qquad K_n(\arg x_n)^2 \to b\,.$$

Moreover,

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_\mathrm{w}} \omega_\mathbb{T} \qquad \Longleftrightarrow \qquad K_n(\arg x_n)^2 \to \infty.$$

Next we consider the *group of p-adic integers*. Let p be a prime. The group of p-adic integers is

$$\Delta_p := \big\{ (x_0, x_1, \ldots) : x_j \in \{0, 1, \ldots, p-1\} \ \text{for all} \ j \in \mathbb{Z}_+ \big\},$$

where the sum $z := x + y \in \Delta_p$ for $x, y \in \Delta_p$ is uniquely determined by the relationships

$$\sum_{j=0}^{d} z_j p^j \equiv \sum_{j=0}^{d} (x_j + y_j) p^j \qquad \mathrm{mod} \ p^{d+1} \qquad \text{for all} \ d \in \mathbb{Z}_+.$$

For each $r \in \mathbb{Z}_+$, let

$$\Lambda_r := \{ x \in \Delta_p : x_j = 0 \ \text{for all} \ j \leqslant r-1 \}.$$

The family of sets

$$\{ x + \Lambda_r : x \in \Delta_p, \ r \in \mathbb{Z}_+ \}$$

is an open subbasis for a topology on Δ_p under which Δ_p is a second countable compact, totally disconnected Abelian T_0-topological group with identity $e := (0, 0, \ldots)$. Its character group is

$$\widehat{\Delta}_p = \{ \chi_{d,\ell} : d \in \mathbb{Z}_+, \ \ell = 0, 1, \ldots, p^{d+1} - 1 \},$$

where

$$\chi_{d,\ell}(x) := \mathrm{e}^{2\pi i \ell (x_0 + p x_1 + \cdots + p^d x_d)/p^{d+1}}$$

for $x \in \Delta_p$, $d \in \mathbb{Z}_+$, $\ell = 0, 1, \ldots, p^{d+1} - 1$. The compact subgroups of Δ_p are Λ_r, $r \in \mathbb{Z}_+$ (see Hewitt and Ross [36], Example 10.16 (a)).

An extended real-valued measure η on Δ_p is a Lévy measure if and only if $\eta(\{e\}) = 0$ and $\eta(\complement\Lambda_r) < \infty$ for all $r \in \mathbb{Z}_+$.

Since the group Δ_p is totally disconnected, the only quadratic form on $\widehat{\Delta}_p$ is $\psi \equiv 0$, and the function $g_{\Delta_p} : \Delta_p \times \widehat{\Delta}_p \to \mathbb{R}$, $g_{\Delta_p} \equiv 0$ is a Lévy function for Δ_p (see Example 5.3.2.4).

For the proof of a Poisson limit theorem on Δ_p we will use the following lemma.

Lemma 8.4.4 *Let* $(\eta_n)_{n \in \mathbb{Z}_+}$ *be extended real-valued measures on* Δ_p *such that* $\eta_n(\complement\Lambda_r) < \infty$ *for all* $n, r \in \mathbb{Z}_+$. *Then the following statements are equivalent:*

(a) $\eta_n(x + \Lambda_r) \to \eta_0(x + \Lambda_r)$ *as* $n \to \infty$ *for all* $r \in \mathbb{N}$, $x \in \complement\Lambda_r$,

(b) $\displaystyle\int_{\Delta_p} f \, d\eta_n \to \int_{\Delta_p} f \, d\eta_0$ *as* $n \to \infty$ *for all* $f \in C_0(\Delta_p)$.

Proof. By Theorem 8.2.3, (b) is equivalent to

(b$'$) $\eta_n|_{\complement U} \xrightarrow{\tau_w} \eta_0|_{\complement U}$ as $n \to \infty$ for all $U \in \mathfrak{V}(e)$ with $\eta_0(\partial U) = 0$.

Obviously, if $\eta_n|_{\complement U} \xrightarrow{\tau_w} \eta_0|_{\complement U}$ holds for some $U \in \mathfrak{V}(e)$ with $\eta_0(\partial U) = 0$ then $\eta_n|_{\complement V} \xrightarrow{\tau_w} \eta_0|_{\complement V}$ holds for all $V \in \mathfrak{V}(e)$ with $V \supset U$ and $\eta_0(\partial V) = 0$. Since $\{\Lambda_r : r \in \mathbb{N}\}$ is an open neighbourhood basis of e and $\partial\Lambda_r = \emptyset$ for all $r \in \mathbb{Z}_+$, (b$'$) is equivalent to

(b$''$) $\eta_n|_{\complement\Lambda_r} \xrightarrow{\tau_w} \eta_0|_{\complement\Lambda_r}$ as $n \to \infty$ for all $r \in \mathbb{N}$.

For distinct elements $x, y \in \Delta_p$, let $\varrho(x, y)$ be the number 2^{-m}, where m is the least nonnegative integer for which $x_m \neq y_m$. For all $x \in \Delta_p$, let $\varrho(x, x) := 0$. Then ϱ is an invariant metric on Δ_p compatible with the topology of Δ_p (see Theorem 10.5 in Hewitt and Ross [36]). Let

$$d(x, y) := \sum_{k=0}^{\infty} 2^{-k} \mathbb{1}_{\{x_k \neq y_k\}} \qquad \text{for all } x, y \in \Delta_p.$$

Then d is a metric on Δ_p equivalent to ϱ, since

$$\varrho(x, y) \leqslant d(x, y) \leqslant 2\varrho(x, y) \qquad \text{for all } x, y \in \Delta_p.$$

Hence the original topology of Δ_p and the topology on Δ_p induced by the metric d coincide. Then weak convergence of bounded measures on the locally compact group Δ_p can be considered as weak convergence of bounded measures on the metric space Δ_p equipped with the metric d.

We show that the set

$$M := \{\mathbb{1}_{x + \Lambda_c} : c \in \mathbb{N}, \ x \in \Delta_p\}$$

is convergence determining for the weak convergence of probability measures on Δ_p. For this one can check that Proposition 4.6 in Ethier and Kurtz [19] is applicable with the following choices: $S := \Delta_p$ equipped with the metric d, S_k is the set $\{0, 1, \ldots, p-1\}$ for all $k \in \mathbb{N}$, d_k is the discrete metric on S_k, $k \in \mathbb{N}$, and

$$M_k := \{f_{c_k} : c_k \in S_k\}, \quad k \in \mathbb{N},$$

where

$$f_{c_k}(x) := \begin{cases} 1 & \text{if } x = c_k, \\ 0 & \text{if } x \neq c_k, \end{cases} \quad x \in S_k, \ k \in \mathbb{N}.$$

For checking we note that for each $c \in \mathbb{N}$ and $x \in \Delta_p$ the function $\mathbb{1}_{x+\Lambda_c}$ is bounded and continuous, since the set $x + \Lambda_c$ is open and closed. Moreover, for each $k \in \mathbb{N}$, S_k with the metric d_k is a complete separable metric space.

It is easy to check that M is also a convergence determining set for the weak convergence of bounded measures on Δ_p. Consequently, (b$''$) is equivalent to

$$(\text{b}''') \quad \int_{\Delta_p} \mathbb{1}_{x+\Lambda_c} \, d\eta_n|_{\complement\Lambda_r} \to \int_{\Delta_p} \mathbb{1}_{x+\Lambda_c} \, d\eta_0|_{\complement\Lambda_r}$$

as $n \to \infty$ for all $x \in \Delta_p$ and for all $c, r \in \mathbb{N}$.

Clearly, this is equivalent to

$$(\text{b}'''') \quad \eta_n\big((x + \Lambda_c) \cap \complement\Lambda_r\big) \to \eta_0\big((x + \Lambda_c) \cap \complement\Lambda_r\big)$$

as $n \to \infty$ for all $x \in \Delta_p$ and for all $c, r \in \mathbb{N}$.

We have

$$(x + \Lambda_c) \cap \complement\Lambda_r = \begin{cases} \Lambda_c \setminus \Lambda_r & \text{if } r \geqslant c \text{ and } x \in \Lambda_c, \\ \emptyset & \text{if } r < c \text{ and } x \in \Lambda_r, \\ x + \Lambda_c & \text{otherwise.} \end{cases}$$

If $r \geqslant c$ then $\Lambda_c \setminus \Lambda_r$ can be written as a union of $p^{r-c} - 1$ disjoint sets of the form $y + \Lambda_r$ with $y \in \Lambda_c \setminus \Lambda_r$. Consequently, (b$''''$) and (a) are equivalent. $\qquad\square$

Theorem 8.2.6 has the following consequence on the group Δ_p of p-adic integers.

Theorem 8.4.5 (Poisson limit theorem)
Let $(X_{n,k})_{n\in\mathbb{N}; \, k=1,\ldots,K_n}$ be a rowwise independent array of random variables in Δ_p. Suppose that there exists a Lévy measure $\eta \in \mathrm{L}(\Delta_p)$ such that

(i) $\displaystyle\max_{1\leqslant k\leqslant K_n} \mathsf{P}\left(((X_{n,k})_0,\ldots,(X_{n,k})_d) \neq 0\right) \to 0$ *as* $n \to \infty$ $\forall d \in \mathbb{Z}_+,$

(ii) $\displaystyle\sum_{k=1}^{K_n} \mathsf{P}\left((X_{n,k})_0 = \ell_0, \ldots, (X_{n,k})_d = \ell_d\right)$

$\to \eta(\{x \in \Delta_p : x_0 = \ell_0, \ldots, x_d = \ell_d\})$ *as* $n \to \infty$

for all $d \in \mathbb{Z}_+$, $\ell_0, \ldots, \ell_d \in \{0, \ldots, p-1\}$ *with* $(\ell_0, \ldots, \ell_d) \neq 0$.

Then the array $\{X_{n,k} : n \in \mathbb{N}, k = 1, \ldots, K_n\}$ *is infinitesimal and*

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_{\mathrm{w}}} \pi_{\eta, g_{\Delta_p}} \qquad as \ \ n \to \infty.$$

Proof. The local mean of any random variable with values in Δ_p is e (with respect to the Lévy function $g_{\Delta_p} \equiv 0$). Moreover, for each $U \in \mathfrak{V}(e)$, there exists $r \in \mathbb{Z}_+$ such that $\Lambda_r \subset U$. Hence, in view of Theorem 8.2.6, it is enough to check that

(i') $\displaystyle\max_{1\leqslant k\leqslant K_n} \mathsf{P}(X_{n,k} \in \complement\Lambda_r) \to 0$ as $n \to \infty$ for all $r \in \mathbb{Z}_+,$

(ii') $\displaystyle\sum_{k=1}^{K_n} \mathsf{E}(f(X_{n,k})) \to \int_{\Delta_p^\times} f \, \mathrm{d}\eta$ as $n \to \infty$ for all $f \in C_0(\Delta_p)$.

Clearly $\{x \in \Delta_p : (x_0, x_1, \ldots, x_d) \neq 0\} = \complement\Lambda_{d+1}$, hence (i') and (i) are identical. Applying Lemma 8.4.4 for $\eta_n := \sum_{k=1}^{K_n} \mathsf{P}_{X_{n,k}}$ and $\eta_0 := \eta$, we conclude that (ii'') and (ii) are equivalent. $\qquad\square$

Remark 8.4.6 Theorem 8.3.4 has the following consequence on Δ_p. If $x_n \in \Delta_p$, $n \in \mathbb{N}$ such that $x_n \to e$, and $(X_{n,k})_{n\in\mathbb{N};\, k=1,\ldots,K_n}$ is a rowwise i.i.d. array of random variables in Δ_p such that $K_n \to \infty$ and $\mathsf{P}(X_{n,k} = x_n) = \mathsf{P}(X_{n,k} = -x_n) = \frac{1}{2}$, then the array $(X_{n,k})_{n\in\mathbb{N};\, k=1,\ldots,K_n}$ is infinitesimal and

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_{\mathrm{w}}} \varepsilon_e \,.$$

Finally we consider the *p-adic solenoid*. Let p be a prime. The p-adic solenoid is a subgroup of $\mathbb{T}^\infty$, namely,

$$S_p = \left\{(y_0, y_1, \ldots) \in \mathbb{T}^\infty : y_j = y_{j+1}^p \ \text{ for all } \ j \in \mathbb{Z}_+\right\}.$$

This is a second countable compact Abelian T_0-topological group with identity $e := (1, 1, \ldots)$. Its character group is

$$\widehat{S}_p = \{\chi_{d,\ell} : d \in \mathbb{Z}_+, \ \ell \in \mathbb{Z}\},$$

where

$$\chi_{d,\ell}(y) := y_d^\ell, \qquad y \in S_p, \quad d \in \mathbb{Z}_+, \quad \ell \in \mathbb{Z}.$$

The set of all quadratic forms on $\widehat{S}_p$ is $\{\psi_b : b \in \mathbb{R}_+\}$, where

$$\psi_b(\chi_{d,\ell}) := \frac{b\ell^2}{p^{2d}}, \qquad d \in \mathbb{Z}_+, \quad \ell \in \mathbb{Z}, \quad b \in \mathbb{R}_+ .$$

An extended real-valued measure η on S_p is a Lévy measure if and only if

$$\eta(\{e\}) = 0 \quad \text{and} \quad \int_{S_p^\times} (\arg y_0)^2 \, \eta(\mathrm{d}y) < \infty.$$

The function $g_{S_p} : S_p \times \widehat{S}_p \to \mathbb{R}$,

$$g_{S_p}(y, \chi_{d,\ell}) := \frac{\ell h(\arg y_0)}{p^d}, \qquad y \in S_p, \quad d \in \mathbb{Z}_+, \quad \ell \in \mathbb{Z},$$

is a Lévy function for S_p.

Theorem 8.2.6 has the following consequence on the p-adic solenoid S_p.

Theorem 8.4.7 (Gauss–Poisson limit theorem)
Let $(X_{n,k})_{n \in \mathbb{N}; \, k=1,\dots,K_n}$ be a rowwise independent array of random variables in S_p. Suppose that there exists a quadruplet $(\{e\}, a, \psi_b, \eta) \in \mathcal{P}(S_p)$ such that

(i) $\displaystyle \max_{1 \leqslant k \leqslant K_n} \mathsf{P}(\exists \, j \leqslant d : |\arg((X_{n,k})_j)| > \varepsilon) \to 0 \quad as \ n \to \infty$
   for all $d \in \mathbb{Z}_+$, $\varepsilon > 0$,

(ii) $\displaystyle \exp\left\{ \frac{\mathrm{i}}{p^d} \sum_{k=1}^{K_n} \mathsf{E}(h(\arg((X_{n,k})_0))) \right\} \to a_d \quad as \ n \to \infty \ \ for \ all \ d \in \mathbb{Z}_+,$

(iii) $\displaystyle \sum_{k=1}^{K_n} \mathsf{V}(h(\arg((X_{n,k})_0))) \to b + \int_{S_p^\times} h(\arg(y_0))^2 \, \eta(\mathrm{d}y) \quad as \ n \to \infty,$

(iv) $\displaystyle \sum_{k=1}^{K_n} \mathsf{E}(f(X_{n,k})) \to \int_{S_p^\times} f \, \mathrm{d}\eta \quad as \ n \to \infty \ \ for \ all \ f \in C_0(S_p).$

Then the array $(X_{n,k})_{n \in \mathbb{N}; \, k=1,\dots,K_n}$ is infinitesimal and

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\ \tau_{\mathrm{w}}\ } \varepsilon_a * \gamma_{\psi_b} * \pi_{\eta, \, g_{S_p}} \qquad as \ n \to \infty.$$

If the limit measure has no compound Poisson factor $\pi_{\eta, \, g_{S_p}}$ then the truncation function h can be omitted. The proof can be carried out as in case of Theorem 8.4.2.

Theorem 8.4.8 (CLT)

Let $(X_{n,k})_{n\in\mathbb{N};\, k=1,\ldots,K_n}$ *be a rowwise independent array of random variables in* S_p. *Suppose that there exist an element* $a \in S_p$ *and a nonnegative real number* b *such that*

(i) $\exp\left\{\dfrac{\mathrm{i}}{p^d} \displaystyle\sum_{k=1}^{K_n} \mathsf{E}(\arg((X_{n,k})_0))\right\} \to a_d$ *as* $n \to \infty$ *for all* $d \in \mathbb{Z}_+$,

(ii) $\displaystyle\sum_{k=1}^{K_n} \mathsf{V}(\arg((X_{n,k})_0)) \to b$ *as* $n \to \infty$,

(iii) $\displaystyle\sum_{k=1}^{K_n} \mathsf{P}(\exists\, j \leqslant d : |\arg((X_{n,k})_j)| > \varepsilon) \to 0$ *as* $n \to \infty$

for all $d \in \mathbb{Z}_+$, $\varepsilon > 0$.

Then the array $(X_{n,k})_{n\in\mathbb{N};\, k=1,\ldots,K_n}$ *is infinitesimal and*

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_{\mathrm{w}}} \varepsilon_a * \gamma_{\psi_b}.$$

Theorem 8.3.4 has the following consequence on S_p.

Theorem 8.4.9 (Limit theorem for rowwise i.i.d. Rademacher array)

Let $(x^{(n)})_{n\in\mathbb{N}}$ *be a sequence in* S_p *with* $x^{(n)} \to e$. *Let* $(X_{n,k})_{n\in\mathbb{N};\, k=1,\ldots,K_n}$ *be a rowwise i.i.d. array of random variables in* S_p *such that* $K_n \to \infty$ *and*

$$\mathsf{P}(X_{n,k} = x^{(n)}) = \mathsf{P}(X_{n,k} = -x^{(n)}) = \frac{1}{2}.$$

Then the array $(X_{n,k})_{n\in\mathbb{N};\, k=1,\ldots,K_n}$ *is infinitesimal.*

If b *is a nonnegative real number then*

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_{\mathrm{w}}} \gamma_{\psi_b} \qquad \Longleftrightarrow \qquad K_n\big(\arg(x_0^{(n)})\big)^2 \to b.$$

Moreover,

$$\mathsf{P}_{\sum_{k=1}^{K_n} X_{n,k}} \xrightarrow{\tau_{\mathrm{w}}} \omega_{S_p} \qquad \Longleftrightarrow \qquad K_n\big(\arg(x_0^{(n)})\big)^2 \to \infty.$$

Appendices

Appendices on topological groups, topological vector spaces and on commutative Banach algebras are added in order to provide the reader with the necessary prerequisites from these topics to be employed throughout the book. The exposition of basic notions and facts from functional analysis organized along specific references and presented in a unified terminology is intended to facilitate the reading of the text.

There are excellent textbooks and monographs available which contain the knowledge layed out in the appendices. The citations on topological groups are justified in chapter II of Volume I of [36] (Hewitt, Ross). The quoted material on topological vector spaces is contained in the graduate text [78] (Schaefer). A systematic treatment of the facts collected on commutative Banach algebras is given in chapters IV and V of the monograph [55] (Loomis). Only a few very special references are documented in the main text of the book.

A   Topological groups

A group G (written additively with neutral element 0) which is also a topological space is said to be **topological group** a if the mapping $(x, y) \mapsto x - y$ from $G \times G$ into G is continuous.

Clearly the translates $x \mapsto a + x$ and $x \mapsto x + b$ for $a, b \in G$ as well as the inversion $x \mapsto -x$ are homeomorphisms from G onto G. As a consequence one notes that for open sets A, B of G also the sets $A + B$, $B + A$ and $-A$ are open. If A is closed and B is compact, then $A + B$ is compact in G.

Let $\mathfrak{V}(x)$ denote the neighbourhood filter of $x \in G$ with the abbreviation $\mathfrak{V}$ for $\mathfrak{V}(0)$. Then $\mathfrak{V}(a) = a + \mathfrak{V} = \mathfrak{V} + a$ whenever $a \in G$.

Theorem A.1 (Determination of the topology of a topological group)

(i) *Let* G *be a topological group with neighbourhood filter* $\mathfrak{V}$. *Then*

    (1) *for every* $U \in \mathfrak{V}$ *there exists* $V \in \mathfrak{V}$ *such that* $V + V \subset U$.

    (2) *If* $U \in \mathfrak{V}$, *then* $-U \in \mathfrak{V}$.

    (3) $0 \in U$ *for all* $U \in \mathfrak{V}$.

    (4) *If* $U \in \mathfrak{V}$, $a \in G$, *then* $a + U - a \in \mathfrak{V}$.

(ii) *Let* G *be a group and let* $\mathfrak{V}$ *be a filter in* G *satisfying the conditions* (1) *to* (4) *of* (i). *Then there exists exactly one topology in* G *such that* G *is a topological group and* $\mathfrak{V}$ *is the neighbourhood filter of* 0.

For every $a \in G$ *one has*

$$\mathfrak{V}(a) = a + \mathfrak{V} = \mathfrak{V} + a\,.$$

We note that for every topological group the closed symmetric neighbourhoods of 0 form a fundamental system of neighbourhoods of 0.

Properties A.2 of **subgroups, products** and **quotients** of a topological group.

A.2.1 *For every subgroup* H *of the topological group* G *its closure* $\overline{H}$ *is again a subgroup of* G.

A.2.2 *Let* $(G_i)_{i \in I}$ *be a family of topological groups. The product topology in* $G := \Pi_{i \in I} G_i$ *is compatible with the group structure in the sense that the mapping*

$$((x_i, y_i))_{i \in I} \mapsto (x_i - y_i)_{i \in I}$$

from $\Pi_{i \in I}(G_i \times G_i)$ *into* G *is continuous.*

A.2.3 *Let* G *be a topological group and let* H *be a normal subgroup of* G. *There exists an equivalence relation*

$$xRy \leftrightarrow x - y \in H\,.$$

The group $G/H := G/R$ *is a topological group with respect to the quotient topology induced by the canonical mapping* $G \to G/H$ *in the sense that* $\mathfrak{V}(0 + H)$ *satisfies the conditions* (1) *to* (4) *of Theorem A.1.*

One observes that an open subgroup of a topological group G is closed. Moreover, open *and* closed subgroups H of G can be generated by a symmetric neighbourhood $U \in \mathfrak{V}$ in the form $H = \bigcup_{n \geqslant 1} nU$.

Definition A.3 A topological group G is said to be **compactly generated** if it contains a compact subset F for which the subgroup generated by F coincides with G, i.e.

$$G = [F] := \{0\} \cup \bigcup_{n \geqslant 1} n(F \cup (-F)).$$

For locally compact groups G this property is equivalent to the requirement that there is an open relatively compact subset U (or a neighbourhood $U \in \mathfrak{V}$) such that $G = [U]$.

Properties A.4 of locally compact groups.

A.4.1 *A Hausdorff (topological) group G is locally compact if and only if 0 possesses a compact neighbourhood.*

A.4.2 *Every closed subgroup of a locally compact group is also locally compact.*

A.4.3 *If G is a Hausdorff group and H a subgroup which is locally compact with respect to the relative topology, then H is closed.*

A.4.4 *Let G be a locally compact group and let H be a normal subgroup of G. Then G/H is a Hausdorff locally compact group if and only if H is closed.*

A.4.5 *Let G be a locally compact group and F a compact subset of G. Then there exists an open and closed compactly generated subgroup H of G with $H \supset F$.*

A.4.6 *Let G be a locally compact group, let H be a subgroup of $\mathbb{R}$ furnished with the relative topology, and let φ be a continuous homomorphism from H into G. Then either $\varphi(H)^-$ is a compact Abelian subgroup of G or φ is a topological isomorphism from H onto $\varphi(H)$.*

B   Topological vector spaces

We now turn to the discussion of commutative groups, in particular to **topological vector spaces** which are vector spaces E over $\mathbb{R}$ and at the same time topological spaces such that the mappings $(x, y) \mapsto x + y$ from $E \times E$ into E and $(\lambda, x) \mapsto \lambda x$ from $\mathbb{R} \times E$ into E are continuous.

Examples B.1 of topological vector spaces are **seminormed** vector spaces E in the sense that they admit a seminorm p with the defining properties

(i) $p \geqslant 0$.

(ii) $p(x + y) \leqslant p(x) + p(y)$ for all $x, y \in E$.

(iii) $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{R}$, $x \in E$.

The function ϱ on $E \times E$ given by

$$\varrho(x, y) := p(x - y)$$

for all $x, y \in E$ provides a *quasi-metric* on E.

B.1.1 Every (real) Banach space is a normed vector space, hence the spaces $\mathbb{R}$ and $\mathbb{C}$ are normed vector spaces (over $\mathbb{R}$).

Let A and B be subsets of a vector space E. A is said to **absorb** B if there exists an $\alpha \in \mathbb{R}_+^\times$ such that $B \subset \lambda A$ for all $\lambda \in \mathbb{R}$ with $|\lambda| \geqslant \alpha$. $A \subset E$ is called **absorbing** (radial) if A absorbs every finite subset of E, and **bounded** if it absorbs each neighbourhood of 0. Finally, $A \subset E$ is called **balanced** (circled) if $\lambda A \subset A$ whenever $\lambda \in \mathbb{R}$ with $|\lambda| \geqslant 1$.

Theorem B.2 (Characterization of topological vector spaces by local properties)

(i) *In every topological vector space E there exists a fundamental system $\mathfrak{U}$ of closed neighbourhoods of 0 such that*

   (1) *every $U \in \mathfrak{U}$ is balanced and absorbing.*

   (2) *For every $U \in \mathfrak{U}$ there is a $V \in \mathfrak{U}$ with $V + V \subset U$.*

(ii) *Let E be a vector space and let $\mathfrak{U}$ be a filter base in E satisfying properties (1) and (2) of (i). Then there exists exactly one topology in E compatible with the vector space structure of E and such that $\mathfrak{U}$ is a fundamental system of neighbourhoods of 0.*

For topological vector spaces one introduces linear subspaces, products and quotients in analogy to the corresponding structures for topological groups.

Theorem B.3 (Characterization of finite dimensional vector spaces)

Let E be a Hausdorff topological vector space with $\dim E = d$ $(< \infty)$. Then every linear mapping from $\mathbb{R}^d$ onto E is an isomorphism and a homeomorphism.

In particular every such topological vector space is isomorphic (as a topological vector space) to $\mathbb{R}^d$.

It is a famous result of F. Riesz that a Hausdorff topological vector space E is finite dimensional if and only if E is locally compact.

A topological vector space E is said to be a **locally convex space** if E admits a fundamental system of convex neighbourhoods of 0.

Clearly every seminormed space, hence every Banach space is locally convex.

There is a characterization of locally convex spaces by local properties analogous to Theorem B.2. In fact, the topology of a locally convex space E is determined by a fundamental system of *convex*, symmetric and absorbing closed neighbourhoods of 0.

Let E be an arbitrary vector space, $(p_i)_{i \in I}$ a family Γ of seminorms on E, and let $\mathfrak{U}$ be the system of all sets V of the form

$$\bigcap_{i \in J} \{x \in E : p_i(x) \leqslant \lambda_i\},$$

where J is a finite subset of I, $\lambda_i > 0$ for all $i \in J$. Then $\mathfrak{U}$ is a filter base, every set $\{x \in E : p_i(x) \leqslant \lambda\}$ $(i \in J, \lambda > 0)$ is convex, symmetric and absorbing, hence every $V \in \mathfrak{U}$ has these properties, and consequently there exists exactly one locally convex topology in E such that $\mathfrak{U}$ is a fundamental system of neighbourhoods of 0. This topology is called the **topology defined by the set** Γ **of seminorms on** E and will be denoted by τ_Γ.

Obviously the choice $\Gamma := \{p\}$ yields the topology of the space E seminormed by p.

Theorem B.4 *Every locally convex topology* τ *on the vector space* E *is of the form* τ_Γ *for some family* Γ *of seminorms on* E.

It is a standard procedure to introduce the inductive limit of locally convex vector spaces. A prominent example of a (strict) inductive limit of locally convex vector spaces is the space $E := C^c(X)$ for a locally compact space X. In fact,

$$C^c(X) = \varinjlim_{K \in \mathcal{K}(X)} C^c(X, K)$$

where for each $K \in \mathcal{K}(X)$ the linear subspace $C^c(X, K) := \{f \in E : \operatorname{supp}(f) \subset K\}$ of $C^c(X)$ carries the topology of uniform convergence.

Theorem B.5 (Banach, Hahn)
Let E *be a vector space,* p *a seminorm on* E, M *a linear subspace of* E *and* f *a linear functional on* M.

The following statements are equivalent:

(i) f can be extended to a linear functional $\overline{f}$ on E satisfying

$$|\overline{f}(x)| \leqslant p(x)$$

for all $x \in E$.

(ii)

$$|f(x)| \leqslant p(x)$$

whenever $x \in M$.

Theorem B.6 (Closed graph)
*Let E and F be Banach spaces. Then any linear mapping from E into
F whose graph is a closed subset of $E \times F$ is continuous.*

In the remaining part of this appendix we are collecting useful notions
and results on the *duality of topological vector spaces.*

Let E and F be topological vector spaces, and let $L(E, F)$ denote
the vector space of all continuous linear mappings from E into F. For a
set S of a given non empty family $\mathfrak{S}$ of bounded subsets of E and for
each $V \in \mathfrak{V}_F(0)$ we introduce the set

$$T(S, V) := \{u \in L(E, F) : u(S) \subset V\}\,.$$

$\tau_{\mathfrak{S}}$ denotes the system of all finite intersections of sets of the form $T(S, V)$.
There exists exactly one topology τ on $L(E, F)$ which is compatible
with the vector space structure such that $\tau_{\mathfrak{S}}$ is a fundamental system of
neighbourhoods of 0 for τ.   The topology τ determined by $\tau_{\mathfrak{S}}$ is said
to be the $\mathfrak{S}$-topology on $L(E, F)$.

Discussion B.7 of the $\mathfrak{S}$-topology.

B.7.1 If F is a locally convex space, then the $\mathfrak{S}$-topology is locally
convex.

B.7.2 Let F be a locally convex space, and let Γ denote the set of
seminorms determining the topology of F (see Theorem B.4). For $p \in \Gamma$
and $S \in \mathfrak{S}$ we introduce the seminorm p_S on $L(E, F)$ by

$$p_S(u) := \sup_{x \in S} p(u(x))$$

for all $u \in L(E, F)$.   With $V := \{y \in F : p(y) \leqslant 1\}$ we obtain

$$T(S, V) = \{u \in L(E, F) : p_S(u) \leqslant 1\}\,.$$

Therefore the family $\{p_S : S \in \mathfrak{S}, p \in \Gamma\}$ of seminorms defines an $\mathfrak{S}$-
topology on $L(E, F)$.

By $L_{\mathfrak{S}}(E, F)$ we abbreviate the vector space $L(E, F)$ equipped with the $\mathfrak{S}$-topology.

B.7.3 Let E, F be topological vector spaces, let F be a Hausdorff space, and let $\bigcup_{S \in \mathfrak{S}} S$ be a dense subset of E. Then $L_{\mathfrak{S}}(E, F)$ is Hausdorff.

Specialization B.8 of $\mathfrak{S}$-topologies.

B.8.1 If $\mathfrak{S}$ is the family $\mathcal{F}(E)$ of finite subsets of E, then $\tau_{\mathfrak{S}}$ yields the topology of **simple** (pointwise) **convergence**.

B.8.2 For $\mathfrak{S} := \mathcal{K}(E)$ $\tau_{\mathfrak{S}}$ determines the topology of **compact convergence**.

B.8.3 If $\mathfrak{S}$ is the family $\mathcal{B}(E)$ of bounded subsets of E, then $\tau_{\mathfrak{S}}$ defines the topology of **bounded convergence** (which appears to be the finest of all $\mathfrak{S}$-topologies).

A set $H \subset L(E, F)$ is said to be **equicontinuous** if for every $V \in \mathfrak{V}_F(0)$ there exists a $U \in \mathfrak{V}_E(0)$ such that

$$u(U) \subset V$$

for all $u \in H$, or equivalently if for every $V \in \mathfrak{V}_F(0)$ we have

$$\bigcap_{u \in H} u^{-1}(V) \in \mathfrak{V}_E(0).$$

Properties B.9 of equicontinuous sets.

B.9.1 *For any equicontinuous subset H of $L(E, F)$ the closure $\overline{H}$ of H taken with respect to the topology of simple convergence (and hence with respect to any finer topology) on $L(E, F)$ is also equicontinuous.*

B.9.2 *Every equicontinuous subset H of $L(E, F)$ is bounded with respect to each $\mathfrak{S}$-topology.*

B.9.3 *If E and F are Banach spaces, then every simply bounded subset of $L(E, F)$ is equicontinuous.*

B.9.4 *If E and F are Banach spaces, then $L(E, F)$ is itself a Banach space with respect to the topology of bounded convergence.*

Let F and G be vector spaces, and let B be a bilinear form on $F \times G$. One says that (F, G) forms a **dual pair** with respect to B if for all $x \in F$, $x \neq 0$ there exists a $y \in G$ such that $B(x, y) \neq 0$, and if for all $y \in G$, $y \neq 0$ there exists an $x \in F$ such that $B(x, y) \neq 0$.

If E is a locally convex Hausdorff space and $E' := L(E, \mathbb{R})$ its **topological dual** (space), then (E, E') forms a dual pair with respect to the bilinear form

$$(x, x') \mapsto B(x, x') = \langle x, x' \rangle := x'(x)$$

on $E \times E'$.

For a dual pair (F, G) with respect to a bilinear form B also (G, F) is a dual pair with respect to B. In the sequel we shall employ the notation $(x, y) \mapsto B(x, y) =: \langle x, y \rangle$ for the bilinear forms defining the dual pairs (F, G) and (G, F).

Given a dual pair (F, G) the **weak topology** $\sigma(F, G)$ is introduced on F by the property that all linear functionals $x \mapsto \langle x, y \rangle$ $(y \in G)$ are continuous.

Properties B.10 of the weak topology.

B.10.1 $\sigma(F, G)$ *is determined by the set* $\{p_y : y \in G\}$ *of seminorms* p_y *on* F *given by*

$$p_y(x) := |\langle x, y \rangle|$$

for all $x \in F$.

B.10.2 $\sigma(F, G)$ *is a locally convex Hausdorff topology.*

B.10.3 F *and* $\sigma(F, G)$ *determine* G *within isomorphisms, i.e.* $F' \cong G$, *where the prime refers to the topology* $\sigma(F, G)$.

Let E be a locally convex Hausdorff space with topology τ. Then the **weakened topology** $\sigma(E, E')$ on E is coarser than τ, and E' is also the topological dual of E with respect to $\sigma(E, E')$.

Theorem B.11 (Alaoglu, Bourbaki)
For a locally convex space E *any equicontinuous subset* H *of* E' *is* $\sigma(E', E)$*-relatively compact.*

Let (F, G) be a dual pair of vector spaces. Without loss of generality we assume that $G \subset F^*$, where F^* denotes the algebraic dual of F. A locally convex topology τ on F is said to be **compatible with the duality** if $G = F'$, where the prime refers to the topology τ. In particular, $\sigma(F, G)$ is compatible with the duality.

One shows by employing Mazur's separation theory that all topologies on F compatible with the duality yield the same system of closed convex sets. For each convex subset of F all topologies compatible with the duality lead to the same closure.

Theorem B.12 *Let* (F,G) *be a dual pair.*

(i) *Every locally convex Hausdorff topology* τ *on* F *is compatible with the duality.*

(ii) τ *is the* $\mathfrak{S}$*-topology for a covering* $\mathfrak{S}$ *of* G *by convex, symmetric and* $\sigma(G,F)$*-compact subsets.*

The finest among the $\mathfrak{S}$-topologies of (ii) (called the *Mackey topology* and denoted by $\tau(F,G)$) is defined by the system $\mathfrak{S}$ of all convex, symmetric and $\sigma(G,F)$-compact subsets of G. The coarsest among those $\mathfrak{S}$-topologies is $\sigma(F,G)$.

Theorem B.13 (Arens, Mackey)
A locally convex topology τ *on* F *is compatible with the duality if and only if*

$$\sigma(F,G) \succ \tau \succ \tau(F,G).$$

Application B.14 to the dual pair (E, E') of a Banach space E.
Let

$$H := \{x' \in E' : \|x'\| \leqslant 1\}$$

be the unit ball of E'. *Then*
B.14.1 H *is equicontinuous, hence* $\sigma(E', E)$*-compact.*

In fact, H is $\sigma(E', E)$-relatively compact by Theorem B.11. Moreover, H is easily seen to be $\sigma(E', E)$-closed.
B.14.2 *If* E *is separable, then* H *is* $\sigma(E', E)$*-metrizable.*

One just notes that along with E also E' is separable and that under this hypothesis every equicontinuous subset of E' is metrizable with respect to the topology of simple convergence.
B.14.3 *For every* $x \in E$ *one has*

$$\|x\| = \sup_{\substack{x' \in E' \\ \|x\| \leqslant 1}} |\langle x, x' \rangle|.$$

This identity follows from the fact that the unit ball of E is the polar of the unit ball of E'.

C   Commutative Banach algebras

Let A be a **normed algebra** in the sense that A is an algebra and a normed vector space over $\mathbb{C}$ such that for the norm $\|\cdot\|$ in A the

inequality

$$\|xy\| \leqslant \|x\|\,\|y\|$$

holds whenever $x, y \in A$. If A admits a multiplicative unit 1 then A may be renormed such that $\|1\| = 1$. For the subsequent discussion we assume this renorming being done.

Theorem C.1 (Gelfand, Mazur)
Every commutative normed algebra A which at the same time is a field, is algebraically and topologically isomorphic to $\mathbb{C}$.

Now let A be a commutative Banach algebra with unit 1. An ideal I of A is said to be maximal if $I \neq A$ and if for every ideal I of A with $I \subset J \subset A$ one has either $J = A$ or $J = I$.

Properties C.2 of ideals I of A

C.2.1 $\overline{I}$ *is an ideal of A.*

C.2.2 *If I is maximal then I is closed.*

C.2.3 *For each closed ideal I of A the quotient A/I is a Banach algebra.*

Theorem C.3 (Gelfand)
For each subset I of a commutative Banach algebra with unit 1 the following statements are equivalent:

(i) *I is a maximal ideal of A.*
(ii) *I is a closed maximal ideal of A.*
(iii) *There exists a continuous epimorphism $f : A \to \mathbb{C}$ such that $f^{-1}(0) = I$.*
(iv) *There exists an epimorphism $f : A \to \mathbb{C}$ such that $f^{-1}(0) = I$.*

For the proof of the implication (i)$\Rightarrow$(iii) one observes that by Property C.2.3 A/I is a Banach algebra and by the maximality of I that A/I is a field. But this implies the existence of an algebraic and topological isomorphism $g : A/I \to \mathbb{C}$. Considering the canonical mapping $p : A \to A/I$ and putting $f := g \circ p$ the equalities

$$f^{-1}(0) = p^{-1}(g^{-1}(0)) = p^{-1}(0) = I$$

yield the desired statement.

Preparations C.4

C.4.1 *Every continuous epimorphism $h : A \to \mathbb{C}$ satisfies $\|h\| \leqslant 1$.*

Let $\Delta(A)$ denote the set of all (continuous) epimorphism from A onto $\mathbb{C}$.

C.4.2 $\Delta(A)$ *is a $\sigma(A', A)$-compact subset of the unit ball of A'.*

$\Delta(A)$ is said to be the **maximal ideal space** or the spectrum of A. For every $x \in A$ the mapping $\hat{x} : \Delta(A) \to \mathbb{C}$ defined by

$$\hat{x}(h) := h(x)$$

for all $h \in \Delta(A)$ is continuous, the mapping $x \mapsto \hat{x}$ from A into $C(\Delta(A))$ is a homomorphism, and

$$\|\hat{x}\| = \sup_{h \in \Delta(A)} |\hat{x}(h)| = \sup_{h \in \Delta(A)} |h(x)| \leqslant \|x\|$$

whenever $x \in A$.

C.4.3 $\hat{A} := \{\hat{x} : x \in A\}$ *is a subalgebra of $C(\Delta(A))$ which separates $\Delta(A)$ and contains 1.*

*In general, $\hat{A}$ is **not** closed in $C(\Delta(A))$.*

C.4.4 *The topology of $\Delta(A)$ is the initial topology with respect to the set $\hat{A}$.*

This topology τ_{G} on $\Delta(A)$ is called the **Gelfand topology**. In this connection we also introduce the **Gelfand transform** $\hat{x}$ of $x \in A$ and the **Gelfand mapping** (representation) $x \mapsto \hat{x}$ from A into $C(\Delta(A))$.

Theorem C.5 *Let A be a commutative Banach algebra with unit 1. Then*

(i) *every epimorphism $\hat{h}$ from $\hat{A}$ onto $\mathbb{C}$ is of the form*

$$\hat{h}(\hat{x}) = \hat{x}(h)$$

for some $h \in \Delta(A)$.

(ii) *Every $h \in \Delta(A)$ defines an epimorphism*

$$\hat{x} \mapsto \hat{x}(h)$$

from $\hat{A}$ onto $\mathbb{C}$.

A Banach algebra A is said to **involutive** if A admits an involution $x \mapsto x^{\sim}$ defined as a mapping $A \to A$ with the properties that

(1) $(x + y)^{\sim} = x^{\sim} + y^{\sim}$,
(2) $(\lambda x)^{\sim} = \bar{\lambda} x^{\sim}$,

(3) $(xy)^\sim = y^\sim x^\sim$, and

(4) $x^{\sim\sim} = x$

whenever $x, y \in A$, $\lambda \in \mathbb{C}$.

Moreover, an involutive Banach algebra A is called a C^*-**algebra** if

$$\|xx^\sim\| = \|x\|^2$$

for all $x \in A$.

Theorem C.6 (Gelfand's formula)
Let A be a commutative Banach algebra with unit and Gelfand mapping $x \mapsto \hat{x}$. Then, for each $x \in A$ the formula

$$\|\hat{x}\| = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$$

holds.

Theorem C.7 *Every commutative C^*-algebra A with 1 is **semisimple** in the sense that the Gelfand mapping $x \mapsto \hat{x}$ of A is a norm- and involution preserving isomorphism from A onto $C(\Delta(A))$.*

For the proof one starts by showing that $x \mapsto \hat{x}$ is involution-preserving. Here preparation C.4.1 is applied. The property of $x \mapsto \hat{x}$ being norm-preserving is established with the help of Gelfand's formula C.6.

In fact, we know already that $\|\hat{x}\| \leqslant \|x\|$ holds for all $x \in A$. For the inverse inequality we first choose $x \in A$ with $x = x^\sim$. Then

$$\|x^2\| = \|xx^\sim\| = \|x\|^2,$$

hence $\|x^2\|^{\frac{1}{n}} = \|x\|$, and by induction

$$\|x^{2^n}\|^{\frac{1}{2^n}} = \|x\|$$

whenever $n \geqslant 1$. Now, Theorem C.6 applies and yields

$$\|\hat{x}\| = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \|x^{2^n}\|^{\frac{1}{2^n}} = \|x\|.$$

Next, for arbitrary $x \in A$ we have $(xx^\sim)^\sim = xx^\sim$, hence

$$\|x\|^2 = \|xx^\sim\|$$

and therefore

$$\|x\|^2 = \|xx^\sim\| = \|(xx^\sim)^\wedge\|$$

$$= \|\widehat{x}\widehat{x^\sim}\|$$

$$= \|\hat{x}\hat{\bar{x}}\|$$

$$= \| |\hat{x}|^2 \| = \|\hat{x}\|^2.$$

Finally, $\hat{A}$ being an involutive subalgebra of $C(\Delta(A))$ with unit which separates $\Delta(A)$ is dense in $C(\Delta(A))$. But along with A also $\hat{A}$ is complete, hence closed, and $\hat{A} = C(\Delta(A))$ has been established.

Now, let A be an involutive commutative Banach algebra which does not admit a unit. By $\Delta(A)$ we again denote the set of all continuous epimorphisms from A onto $\mathbb{C}$, furnished with the initial topology with respect to the mappings $h \mapsto h(x)$ from $\Delta(A)$ into $\mathbb{C}$ (for all $x \in A$). Let $\tilde{A} := A \oplus \{1\}$ denote the involutive commutative algebra arising from adjoining a unit element 1. $\tilde{A}$ can be given a norm such that it becomes a Banach algebra with unit 1 and Gelfand mapping $x \mapsto \hat{x}$ (related to $\Delta(\tilde{A})$ and $\widehat{\tilde{A}}$). Let $h_0 \in \Delta(\tilde{A})$ be defined by

$$h_0(x) := \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x = 1. \end{cases}$$

Clearly, $\Delta(\tilde{A}) \setminus \{h_0\}$ is a locally compact space, and the mapping $h \mapsto \mathrm{Res}_A\, h$ is a homeomorphism from $\Delta(\tilde{A}) \setminus \{h_0\}$ onto $\Delta(A)$. Therefore $\Delta(A)$ is locally compact, and $x \in \tilde{A}$ belongs to A if and only if $\hat{x}$ (restricted to $\Delta(A)$) vanishes at infinity.

Theorem C.8 *Let A be an involutive commutative Banach algebra. Then*

(i) *$\hat{A}$ is a subalgebra of $C^0(\Delta(A))$.*

(ii) *If, in addition, A is a C^*-algebra, then A is **semisimple** in the sense that $x \mapsto \hat{x}$ is a norm- and involution preserving isomorphism from A onto $C^0(\Delta(A))$.*

While (i) is clear by the remarks preceding the theorem, only (ii) requires an argument. In fact, there exists a unique norm on A which extends to $\tilde{A}$ and makes $\tilde{A}$ a C^*-algebra. From Theorem C.7 we conclude that $\tilde{A}$ is semisimple. Restricting the Gelfand mapping $x \mapsto \hat{x}$ to A and $C^0(\Delta(A))$ respectively we reach the desired conclusion.

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Symbols

$|A|$ . . . . cardinality of A

$\complement A$ . . . . $= \complement_E A$ complement of A in E

A^1 . . . . set of normed elements of A

A_+ . . . . $= \{a \in A : a \geqslant 0\}$

$A^\times$ . . . . set of elements of A different from a distinguished element

A^{b} . . . . set of bounded elements of A

$A_{\mathbb{R}}$ . . . . set of real elements of A

$A_{\mathbb{C}}$ . . . . set of complex elements of A

$[A]$ . . . . group generated by a set of A

$[A]^-$ . . . . closed group generated by a set of A

$\langle A \rangle$ . . . . semigroup generated by a set of A

$\langle A \rangle^-$ . . . . closed semigroup generated by a set of A

A^ε . . . . $= \{x \in E : d(x, A) < \varepsilon\}$ $(\varepsilon > 0)$

$\mathbb{1}_A$ . . . . indicator of a set A

Res_A . . . . restriction to a set A

$\mathbb{N}$ . . . . set of positive integer numbers

$\mathbb{Z}$ . . . . group of integer numbers

$\mathbb{T}$ . . . . torus

$\mathbb{C}$ . . . . field of complex numbers

$\mathbb{Q}$ . . . . field of rational numbers

$\mathbb{Q}_{\mathrm{d}}$ . . . . discretely topologized $\mathbb{Q}$

$\mathbb{Q}_p$ . . . . group of p-adic numbers $(p > 1)$

$\mathbb{R}$ . . . . field of real numbers

$\overline{\mathbb{R}}$ . . . . $= \mathbb{R} \cup \{-\infty, \infty\}$

$\mathbb{R}^d$ . . . . d-dimensional Euclidean space

$\mathbb{Z}^d$ . . . . d-dimensional lattice

$\mathbb{T}^d$ . . . . d-dimensional torus

$\mathbb{C}^{m \times n}$ . . . . set of $m \times n$-matrices over $\mathbb{C}$

Re . . . . real part

Im . . . . imaginary part

$\wedge$ . . . . minimum

$\vee$ . . . . maximum

$[\mu]$ . . . . μ-almost everywhere

$\mathfrak{V}(x)$ . . . . neighbourhood filter of x (for any topological space)

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